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ESTIMATION OF A POSTERIORI COMPUTATIONAL ERROR BY THE HIGHER ORDER MULTIPOINT MESHLESS FDM

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Abstract. The main objective of this paper is to present a possibility of high quality a posteriori error evaluation based on reference solutions obtained by means of the new multipoint meshless finite difference method. Due to its higher order approximation, the multipoint results may be used as improved reference solution instead of the true analytical one to estimate the errors. Several types of a posteriori error estimators which can be used to evaluate the calculation error are described here. The results of selected numerical benchmarks are considered.

Keywords: Error analysis, meshless FDM, multipoint method, higher order approximation

Mathematics Subject Classification 2010: 65-G99, 65-M99, 68-U20, 65-D99

1 INTRODUCTION

The error estimation of numerical solution of the boundary value problems constitutes an important part of computational analysis. In general, the analytical solutions of partial differential equations which describe the mathematical model of an engineering problem hardly exist. Therefore, numerical methods, including the meshless ones, are employed to evaluate the solution accuracy.

Basically, two types of error estimation procedures are available. A priori error estimates the asymptotic behavior of the discretization errors. It is usually applied after discretization process, before the whole solution process starts. It estimates the solution convergence rate by using only mesh modulus h and approximation order p, as well as the basic mathematical foundations. A priori error might be very effective, if it is applied to regular meshes and to simple linear differential operators.

A posteriori error estimators [1], instead, may be evaluated only after the problem is solved, and are designed to provide good approximation of the solution errors for a given discretization. This type of errors is often used in adaptive schemes application where the mesh is locally or globally refined. Nowadays a posteriori error analysis and effective error estimation still are one of the most important problems in the discrete analysis, especially in real engineering tasks.

Many strategies have been developed to estimate the a posteriori errors in the most accurate manner. For this purpose, in order to evaluate the error of the calculated numerical results, an improved reference solution is used instead of the true analytical one. Two mechanisms may be applied in order to obtain an improved solution. The first one is based on the mesh density increase, preferably using an adaptive (h-type) solution approach. The second mechanism is provided by rising the approximation order (p-type). A mixture of these both ways may be also applied.

Among various high order (HO) approximation approaches, the multipoint meshless finite difference method (MFDM) [8, 9] may be used, as shown below.

The main objective of this paper is to present a possibility of high quality a posteriori error evaluation based on reference solutions obtained by means of the multipoint MFDM.

The concept of the HO multipoint approach is based on raising the approximation order of searched function by using a combination of its values together with a combination of additional degrees of freedom at all FD star (stencil) nodes. The known values of right hand side of the considered differential equation or the other chosen operators may be used as the additional d.o.f. This improves approximation order and consequently the solution without increasing the number of nodes in the mesh. Moreover, the higher order FD operator is generated using the same set of nodes in FD star as in the standard meshless FDM case. This fact is an advantage when compared with the other HO methods (e.g. so called defect/deferred correction approach [4], based on increasing the number of nodes included into the stencil) because of less calculations needed.

Due to its higher order approximation quality (Figure 1), the multipoint MFD results may be applied for two purposes [1, 9, 12, 14]: to examine the solution quality, and based on it generating a series of adaptive meshes.

In this paper we review the basic concepts applied to evaluation of a posteriori error estimates using the high quality HO reference solutions calculated by the multipoint MFDM.

The outline of this paper is as follows: next chapter shortly presents the concept of the multipoint meshless method and its two main approaches for various formulations of the boundary value problems. The basic ideas to establish local error estimate as well as the global ones are given in the following chapter. Several types of a posteriori error estimators which can be used to evaluate the calculation error, and some error indicators useful for the adaptive solution approach are introduced next. Furthermore, the results of selected numerical benchmarks are considered, eventually short summary and some final remarks are given.

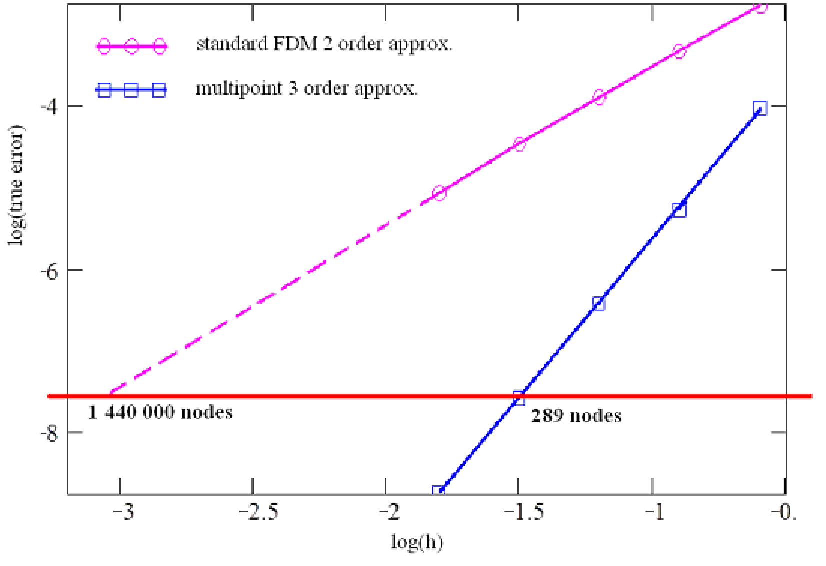

Figure 1. Solution convergence results for both the higher order multipoint approach and the standard FDM; the simple test of Poissons b.v. problem [7]

2 MULTIPOINT APPROACH AS THE REFERENCE SOLUTION

The idea of the multipoint technique was developed long time ago by Collatz [3]. It is based on introducing additional degrees of freedom (e.g. known right hand side values of the considered differential equation) at all FD star nodes in order to obtain the higher order FD operator.

The original multipoint Collatz concept has been recently modified and extended by the authors to the new fully automatic multipoint meshless FDM [8]. For this purpose, the multipoint method is based on the moving weighted least squares (MWLS) approximation technique [13] instead of the polynomial interpolation, as proposed by Collatz. Moreover, unstructured, arbitrarily distributed clouds of nodes [11, 13] may be applied here rather than regular meshes only. Besides development of the multipoint MFDM for the analysis of b.v. problems given in the local (strong) formulation, the method was also extended to the global and global-local formulations including the minimum of the total potential energy, the variational Galerkin and the meshless local Petrov-Galerkin (MLPG) ones.

In the multipoint MFDM may be applied any formulation involving unknown function and its derivatives.

2.1 Boundary Value Problem Formulations

Let us consider the local (strong) formulation of a boundary value problem given in a domain Ω for the n^{th} order ODE (PDE) with appropriate b.c.

$$\begin{cases} \mathcal{L}u = f, & \\ \mathcal{G}u = g, & \end{cases} \qquad u = u(P), \qquad \begin{array}{l} \text{for } P \in \Omega, \\ \text{for } P \in \partial\Omega \end{array} \tag{1}$$

or an equivalent global (weak) one involving integral of the type

$$\int_{\Omega} F(u,\, \mathcal{L}u)\, \mathrm{d}\Omega \tag{2}$$

where $\mathcal{L}$, $\mathcal{G}$ are differential operators.

The global (weak) formulation may be posed in the domain Ω either in the form of:

- a functional optimisation (e.g. minimization of the potential energy functional)

$$\min_{u} I(u), \quad I(u) = \frac{1}{2} b(u,u) - l(u) \tag{3}$$

- or more general, as variational principle

$$b(u,v) = l(v), \quad \forall v \in V \tag{4}$$

  together with given equality and inequality constraints, where b – is a bilinear functional dependent on the trial function u and test function v, V – is the space of the test function, l – is a linear operator dependent on v. In both cases corresponding boundary conditions have to be satisfied.

In the variational formulations we may deal with the Petrow-Galerkin approach when u and v are different functions from each other, and Bubnov-Galerkin one when u and v are the same. Moreover, the variational form may have symmetric or non-symmetric character.

More and more frequently are recently used so called meshless local Petrov-Galerkin formulations (MLPG), especially for the meshless methods. In this case, the whole domain Ω may be divided into a finite number of subdomains, usually assigned to each selected node. Assuming a test function v "locally defined" on each subdomain Ω_i (assume to be equal to zero elsewhere except the area of Ω_i) one may obtain a global-local formulation of the Petrov-Galerkin type. The original functional minimisation or variational principle is practically applied rather to those local subdomains than to the whole domain at once then.

The test function v may be defined in various ways. In particular, in the most effective version, MLPG5 formulation [2] – the Heaviside type test function

$$v = \begin{cases} 1, & \text{in } \Omega_i, \\ 0, & \text{outside } \Omega_i \end{cases} \tag{5}$$

is assumed. Hence any derivative of v is also equal to zero in the whole domain Ω. Therefore, relevant expressions in the functional $b(u, v)$ and in $l(v)$ vanish, reducing in this way amount of calculations involved.

All variants of the global (weak) formulation of the multipoint method may be realized using regular or totally irregular meshes, like in the case of the local formulation of b.v. problems.

2.2 Multipoint Method

Using the FDM or MFDM discretization based on the selected FD stars (Figure 2) with respect to the central node i, the classical difference operator Lu is presented in the following form:

$$\mathcal{L}u_i \approx Lu_i \equiv \sum_j c_j u_j = f_i \quad \Rightarrow \quad Lu_i = f_i, \quad c_j = c_{j(i)}. \tag{6}$$

However, in the multipoint formulation, the difference operators L and M are obtained by means of the Taylor series expansion of unknown function u and additional degrees of freedom. For this purpose e.g. a combination of the known right hand side values f of the considered equation at each node of FD star is applied instead of the function value at the central node only (Figure 2):

$$\mathcal{L}u_i \approx Lu_i \equiv \sum_j c_j u_j = \sum_j \alpha_j f_j \quad \Rightarrow \quad Lu_i = Mf_i. \tag{7}$$

Here, j – is a number of a node in the selected FD star, Mf_i – is a combination of the equation right hand side nodal values, f_j – may present value of the whole operator $\mathcal{L}$ or its part only, e.g. a specific derivative $u_j^{(k)}$.

Two basic versions of the multipoint MFDM are considered: the general and the specific ones [7, 8]. Equation (7) presents the basic formula for the *specific* multipoint formulation.

In the *general* multipoint form, selected derivative $u^{(k)}$ is used as additional d.o.f. rather then the right hand side f of the given differential equation. The multipoint formula is as follows:

$$\sum_j c_j u_j = \sum_j \alpha_j u_j^{(k)}. \tag{8}$$

Application of the specific approach is simpler and easier in implementation, but is mainly restricted to the linear b.v. problems. When the specific formulation cannot

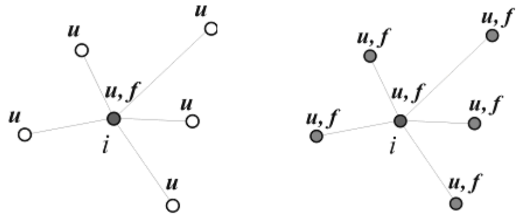

Figure 2. MFD stars used in MFDM and the multipoint approach

be applied (e.g. for nonlinear b.v. problems), more complex various versions [7] of the general approach may be used. In each of these multipoint MFD cases, one may obtain higher order approximation, using the same FD star, as needed to generate FD operators in the classical MFDM approach.

Function u_j and additional d.o.f. of such right hand side f_j (or derivative $u_j^{(k)}$ in the general case) are developed into the truncated Taylor series including higher order derivatives with the respect to a chosen central point P_i.

$$u_j = \bar{u}_j + R_{0j}, \quad f_j = \bar{f}_j + R_j. \tag{9}$$

Afterwards, the weighted error functional

$$J_i = \sum_j (u_j - \bar{u}_j)^2 \, {}^0w_{ij}^2 + \sum_j (f_j - L\bar{u}_j)^2 \, {}^nw_{ij}^2 \tag{10}$$

for higher order approximation of u_j is generated. Here, the weighting factor w is a function of the distances $\rho_{ij} = |P_i - P_j|$ between points P_i and P_j, n – is the order of differential equation.

One of the following weighting factors is assumed [19]:

$${}^kw_{ij}^2 = \rho_{ij}^{2(-p+k-1)}, \quad \text{or} \quad {}^kw_{ij}^2 = \left(\rho_{ij}^2 + \frac{g^4}{\rho_{ij}^2 + g^2}\right)^{-p+k-1} \tag{11}$$

where p – is the local approximation order, $g >= 0$ – is a smoothing parameter.

Minimization of the functional $\partial J/\partial\{D\mathbf{u}\} = \mathbf{0}$ yields at each node i the local multipoint MFD formulas for $q = 0, 1, \ldots, p$ order derivatives $D\mathbf{u}_i = \left\{u_i, u_i', u_i'', \ldots, u_i^{(p)}\right\}$ first, and finally for the basic equation:

$$D\mathbf{u}_i = \sum_j c_j u_j - \sum_j \alpha_j f_j \tag{12}$$

in the specific case, or type (8) in the general multipoint case.

Having found in the whole domain Ω the FD relations for all derivatives emerging in the b.v.p. equations, we apply these formulas to the problem considered. After such discretization PDE depend on the primary unknowns u only. Collocation approach, carried out in the whole domain $\Omega \cup \partial\Omega$ may be used then in order to generate the simultaneous FD equations and provide the final solution in the case of the local b.v. problem formulation.

Though in all global and global-local formulations of the multipoint MFDM, the function u and its derivatives are always approximated by means of the multipoint finite difference formulas, assumption and discretization of the test function v and its derivatives may be done in many ways. After necessary numerical integration involved e.g. by means of the Gaussian quadrature, the system of discrete simultaneous equations is received.

3 A POSTERIORI ERROR ESTIMATION

As mentioned above, the main purpose of any a posteriori error estimation is to obtain a reliable evaluation of accuracy of the computed numerical solution. Though the exact solution is usually unknown, the higher precision result can be used as the reference solution to evaluate the true error. Due to high quality of results obtained by means of the multipoint approach we may also use them in order to provide reference solutions needed for the global or local error estimation.

When presenting a posteriori error estimators, we use three types of solution:

- u – unknown true analytical solution usually known only for a benchmark problems;
- u^L – rough numerical solution obtained by the standard MFDM or FEM;
- u^H – improved higher order solution usually assumed as the reference one. For this purpose the multipoint HO solution may be used.

We distinguish here:

- local estimation (at any required point) of the solution and/or residual errors;
- global estimation (over a selected subdomain or the whole domain of the considered problem) of the solution and residual errors.

The local error estimation at any required point of the domain or in its boundary is typical for the MFDM. On the other hand, the most commonly used are the global error estimators evaluated over the whole domain or over a chosen subdomain. In the meshless methods, e.g. in two-dimensional approach, it may be applied to such subdomain as the Voronoi polygon or the Delaunay triangle. The global estimators provide information whether the specified subdomain (or the whole domain) mesh needs refinement or raising the approximation order as well as the general information about the accuracy of the computed numerical solution in a chosen suitable norm.

Most of a posteriori error estimators (especially in the FEM) focused on the global error in the energy norm. The energy norm associated with the bilinear form is defined as follows:

$$\|e\|_E = \sqrt{\frac{1}{\Omega}\int_\Omega b(u-\bar{u},\ u-\bar{u})\,\mathrm{d}\Omega}. \tag{13}$$

The global error may be also computed as the discrete $l^2(\Omega)$ mean square norm

$$\|e\|_2 = \sqrt{\frac{1}{N}\sum_i (u_i-\bar{u}_i)^2} \tag{14}$$

as well as the maximum norm. Here u is the true solution and $\bar{u}$ is an approximate rough one.

3.1 Multipoint Based Error Analysis

The special emphasis in this work are focused on a posteriori global and/or local solution errors estimation, as well as the on residual errors both based on the multipoint meshless method.

A posteriori error analysis based on the multipoint MFDM is discussed for any type of formulation of boundary value problems and for discretization using arbitrary irregular clouds of nodes. Several types of error estimations, which were developed for the FEM analysis also, are considered.

Hierarchic Estimators

The hierarchic estimators are based on the comparison of obtained results with reference solution calculated using:

- h-type approach – mesh is locally or globally refined by density increase from h to e.g. $h/2$,
- p-type approach – the approximation order is raised from p to $p+1$, or
- hp-approach – mixed of the above versions.

The higher order multipoint method may be successfully applied as p- or hp-approach to compute improved solution (Figures 2, 3 and 4). It is worth mentioning that the multipoint approach in p-type refinement may provide not only approximation order $p+1$, but may well overcome this value (Figure 3).

In what follows we will use next notations of errors. When the exact analytical solution u of a boundary value problem is known (e.g. in benchmark problems) and rough numerical solution u^L using the standard MFDM approach (6) is found, the true low order solution error is as follows:

$$e = u - u^L. \tag{15}$$

When the multipoint approach (7) is applied, an improved higher order solution u^H is obtained and used as the reference solution. One may then estimate the true solution error (15) in the following way:

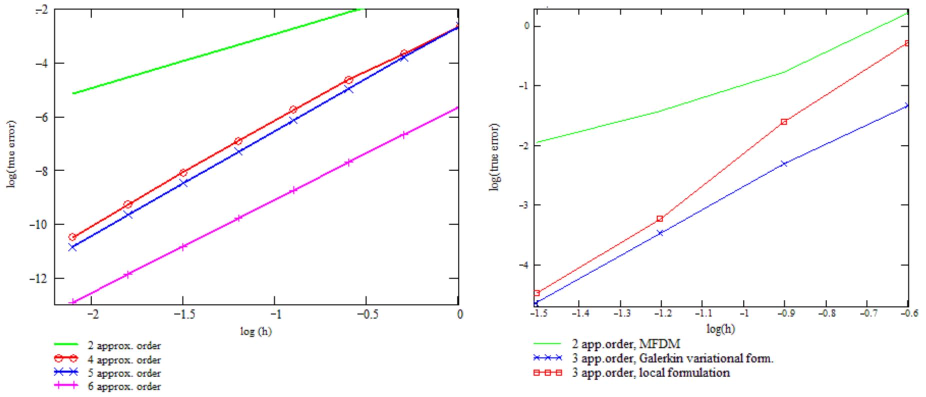

Figure 3. Solution convergence of 1D ($u' + u'' = f$ and b.c.) and 2D (Poissons b.v.p.) tests for various approximation orders

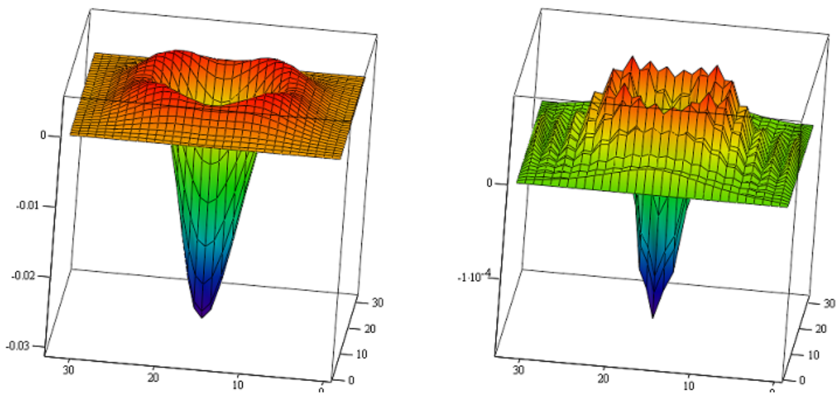

Figure 4. The exact solution error of Poissons b.v.p.: basic MFDM (2nd approx. order, $\max e = 3.0e{-}2$) and multipoint MFDM (3rd approx. order, $\max e = 1.7e{-}4$)

$$\eta = u^H - u^L \approx e. \tag{16}$$

Moreover, the exact higher order solution error

$$e^H = u - u^H \tag{17}$$

may be also estimated when using the multipoint method with various orders of approximation, e.g. $p1$ and $p2 > p1$. In that case

$$\eta^H = u^{H(p1)} - u^{H(p2)} \approx e^H. \tag{18}$$

In order to evaluate the accuracy of error estimation, the so called effectivity index

$$I_{\text{eff}} = 1 + |\|\eta\| - \|e\||/\|e\|, \quad I_{\text{eff}} = 1 + |\eta - e|/e \tag{19}$$

may be used. This index is based on comparison between the true e and estimated η error norms. Ideally its value should be close to 1.0.

Poissons b.v. problem with Dirichlet b.c.	**Test 1**		**Test 2**	
	Max norm	Mean square	Max norm	Mean square
Higher order true error e^H (version 3 local)	1.22e−6	4.12e−7	1.48e−3	5.93e−4
Higher order true error e^H (version 3 global)	1.8e−6	8.29e−7	8.05e−3	3.11e−3
Lower order true error e	2.54e−4	1.24e−4	1.35e−1	2.18e−2
Error estimation η	2.52e−4	1.23e−4	1.27e−1	1.87e−2
Effectivity index I_{eff}	**1.007**	**1.007**	**1.06**	**1.143**

Table 1. Error estimation for various multipoint MFDM versions; regular 249 nodes mesh

Several tests done for the multipoint approach show that the calculated values of the above effectivity index are close to 1 (Table 1), i.e., high quality error estimation was obtained. Here Poissons b.v. problem with Dirichlet b.c. Test 1 corresponds to the exact solution $u(x, y) = \sin(x + y)$ and Test 2 – corresponds to the exact solution $u(x, y) = -x^3 - y^3 + \mathrm{e}^{(x,y)}$ given in the rectangular domain [7].

Residual Estimators

The residual estimators use either explicit residual errors of low order r, or high order r^H as follows:

$$r = Lu^L - f, \quad r^H = Lu^H - f \tag{20}$$

in locally formulated b.v. problem, and in global formulations:

$$r = b(u^L, v) - l(v), \quad \forall v \in V, \quad r^H = b(u^H, v) - l(v) \tag{21}$$

or equivalent implicit ones. The implicit type residual error (not specified here) needs solution of the boundary value problem (4) with the residuum used as the equation right hand side

$$b\,(e, e) = r. \tag{22}$$

Each of these estimators provides a quality measure of the higher (using multipoint approach) or lower (standard MFDM) order solution error.

When approximated higher order solution u^H is defined at nodes, the local residuum between the nodes may be calculated. At nodes the residual error is equal to zero when the collocation requirement is imposed. In common opinion, at the middle of this distance, between two neighbour nodes the local residuum is

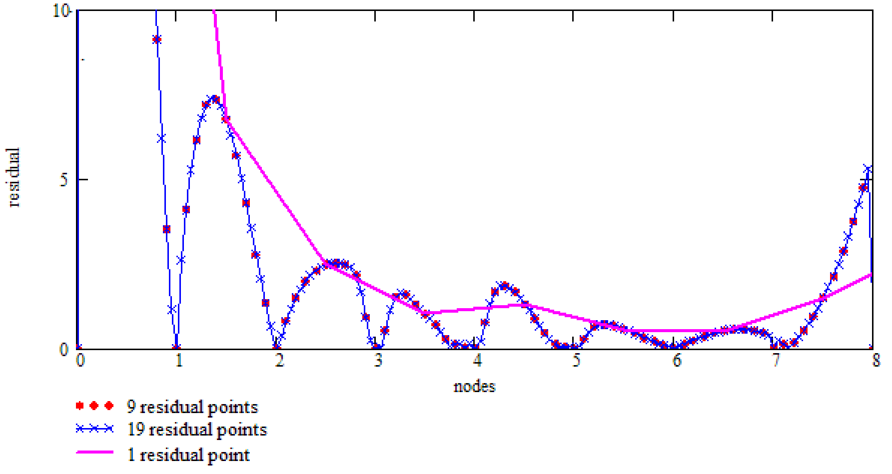

Figure 5. Local residual error distribution using 1, 9 and 19 points between nodes

expected to reach its maximum value. However, several tests done showed that the error distribution (Figures 5, 6) essentially depends on the smoothing parameter g used in the MWLS approximation weight function (11).

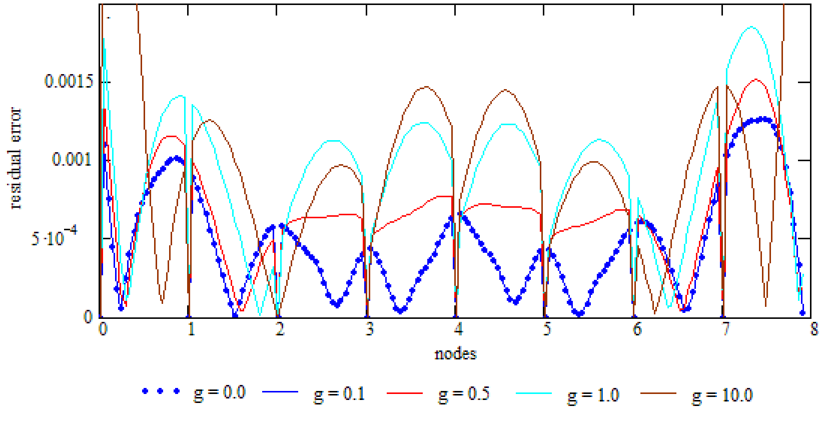

Figure 6. Residual error distribution using 20 points between nodes; the influence of the MWLS weight factor g

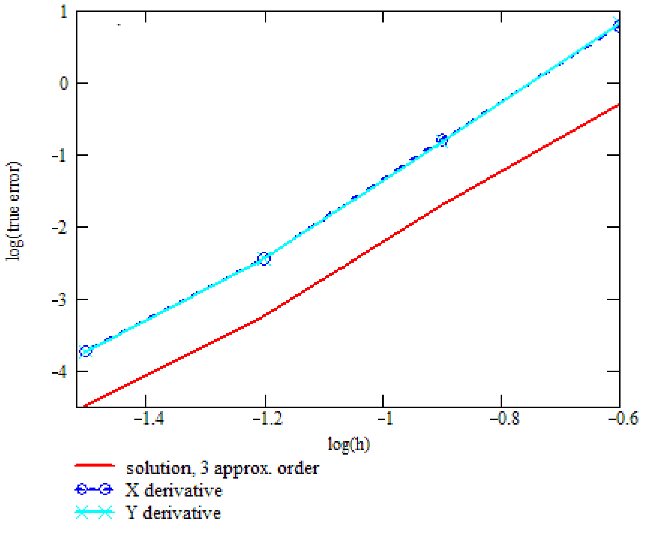

Figure 7. Convergence rates of solution and its derivatives exact errors; Poissons b.v. problem with Dirichlet b.c.

Smoothing Estimators

Smoothing estimators (well known as Zienkiewicz-Zhu one [10, 18]) are based on the difference between the recovered (rough) and the reference (smoothed) derivatives (e.g. stresses) of the solution. The higher order approximation of derivatives up to order p is also obtained when applying the multipoint method. These derivatives may be used instead of the recovered ones in the error estimation process. It is worth noticing here, that beside the improved solution, the multipoint approach provides the evaluation of HO approximation of derivatives without any additional effort. The best feature of the multipoint method is the same order of the convergence rate for derivatives as it is for the solution itself (Figure 7).

3.2 Adaptive Solution Approach

There may be different reasons for using the a posteriori error estimators. Besides the evaluation of the numerical solution, reliable a posteriori estimation techniques may be applied for adaptation strategy controlled by error estimation (especially residual one). It is very effective when mesh generator [11, 15] based on the mesh density control concept is used. The main idea is to avoid abrupt changes in mesh density during mesh generation and modification. The mesh refinement strategy deals with "filters" to the Liszka's sieve method generalized for irregular meshes [11]. These are based upon error indicators calculated as a result of an a posteriori error analysis applied to the considered boundary problem. In general, the new nodes are inserted at points, where admissible error norms are exceeded.

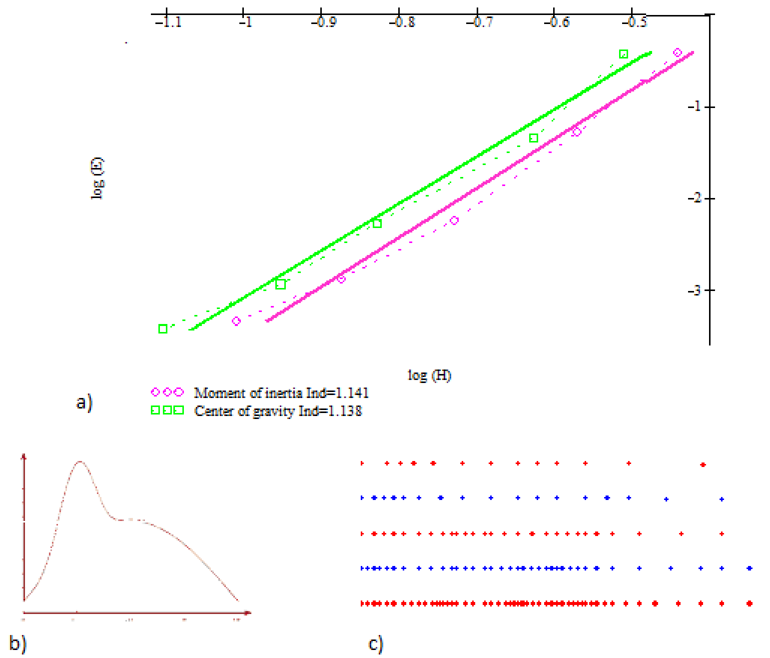

Figure 8. Solution convergence a) of 1D b.v. problem ($u' + u'' = f$ and b.c.); b) based on the series of adaptive meshes; c) using the both types of the error indicators

During the adaptation process the set of strongly irregular meshes is used. Therefore, several global error indicators were proposed to analyze the convergence of the solution and the residual errors. They determine by pair of values which characterize both the measured error e and the domain discretization (mesh modulus h). The indicators allow for effective estimation of the solution and residual convergence on the set of regular and arbitrary irregular adaptive meshes. The evaluation of the error indicator quality is possible by means of the convergence rate a and its mean deviation

$$Ind = \sqrt{\frac{1}{N}\sum_i (e_i - a \cdot x_i - b)^2}. \tag{23}$$

The best results (Figure 8) are obtained for the simplest pairs of the discrete indicators based on the moment of inertia criterion of points (h_i, e_i)

$$\bar{h} = \sqrt{\frac{1}{N}\sum_i h_i^2}, \quad \bar{e} = \sqrt{\frac{1}{N}\sum_i e_i^2} \tag{24}$$

and center of gravity of the scattered data

$$\bar{h} = \frac{1}{N}\sum_i h_i, \quad \bar{e} = \frac{1}{N}\sum_i |e_i|. \tag{25}$$

In the adaptive mesh refinement steer by error estimation only some fundamentals were formulated as well as simple tests were analyzed so far.

3.3 Effective Error Estimator by Means of the Multipoint MFDM

We conclude this chapter with some general remarks on the requirements imposed on the error estimators when using the multipoint MFDM as the reference solution. Following Grätsch and Bathe [5] some features of an effective error estimator have to be reached:

- The *accuracy* of error estimator in the sense that the predicted error should be close to the true (unknown) error.

  In the case of the multipoint approach estimation of the lower order standard MFDM or FEM error is expected to be very good due to higher order approximation applied. The numerical results obtained so far for tested b.v. problems confirm high accuracy of the error estimate.

- The *convergence rate* of the error estimate should be close to the rate of true error and tend to zero when mesh density is raised.

  Using multipoint approach, especially for the fine mesh, the calculated index of effectivity (19) tends to 1.

- The *computational cost* of error estimation has not to be high, when compared to the costs of the total computations of analysis itself.

  In general, the computational cost of the HO meshless error analysis is much smaller when compared with the cost of remeshing or cost of the refinement algorithms commonly applied in the FE analysis. MFDM provides the clear advantage in calculation of derivatives – one can obtain them without any additional effort other than the solution itself.

- The error estimator should be *robust* with regard to a wide range of applications, including nonlinear analysis.

  The numerical results of preliminary tests of application of the multipoint method in the various b.v. problems including nonlinear analysis done so far are very encouraging. The error estimator is expected to be robust for a wide range of applications.

- The *adaptive refinement* process and the mesh optimization, with respect to the purpose of the computation, should be possible when implementing the error estimator.

The multipoint MFDM allows for effective adaptation strategy controlled by the error estimation. So far, only some fundamentals were formulated as well as a number of simple tests was analysed.

- The upper and lower *bounds* of the actual error should be provided by the error estimator.

  Using multipoint method of approximation order p, the mean error bound of solution is expected [3, 8, 17] to be an $O(h^{p+1})$, where h is an average distance between the nodes. Analysis of analytical, a priori error estimate as well as upper and lower bounds is planned in the paper that follows.

4 NUMERICAL RESULTS

Several tests of multipoint MFDM application to error analysis have been solved. Among other Poissons boundary value problems, the prismatic bar twisting [16] was examined. The Prandtl stress function and the shear stress in prismatic bars of various cross-sections like square (for comparison with the true solution), I-beam, and railroad rail shape were analysed. The local formulation of the Saint-Venant problem is as follows

$$\begin{cases} \nabla^2\Phi = -2G\theta, & \text{in } \Omega, \\ \Phi = 0, & \text{on } \partial\Omega \end{cases} \tag{26}$$

where Φ – the Prandtl stress function, $G\theta = 1$ – torsional stiffness, Ω – domain of the bar cross-section. The total shear stress for prismatic bar is

$$\tau = |grad\ \Phi| = \sqrt{\tau_{zx}^2 + \tau_{zy}^2} \tag{27}$$

where shear stresses are

$$\tau_{zx} = \frac{\partial\Phi}{\partial y}, \qquad \tau_{zy} = -\frac{\partial\Phi}{\partial x}. \tag{28}$$

The true solution error analysis for various local and global multipoint meshless FDM formulations are presented in Figures 9 and 10. The Prandtl stress function was obtained for square cross-section where the analytical solution of the Saint-Venant problem is known.

The results of multipoint MFDM error estimations (Figure 11), including estimation of the higher order solution error e^H by using the MLPG5 formulation (Figure 12), are very encouraging. Each formulation of the meshless multipoint method, including the MLPG5 one, provides higher order solution results (in comparison with the standard meshless FDM) of the b.v. problems.

The total shear stress in I-beam and its smoothing error estimation are presented in Figure 13. Finally Figure 14 presents the results obtained for irregular mesh of railroad rail shape bar and its error estimation when compared multipoint 3$^{\text{rd}}$ and 2$^{\text{nd}}$ order approximation.

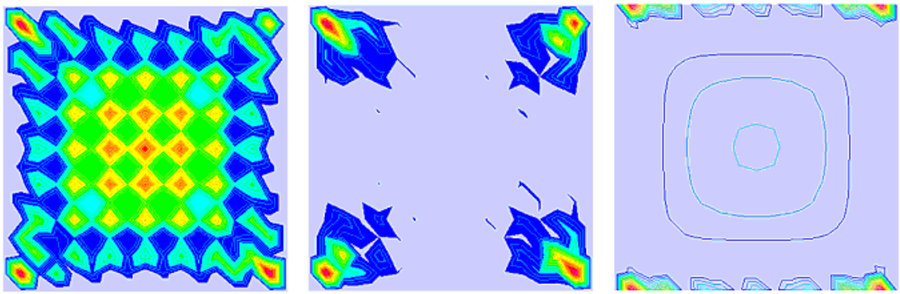

Figure 9. True solution error obtained by the MFDM (max = $8.84e{-4}$), 3rd approx. order multipoint local formulation (max = $5.04e{-4}$), 3rd approx. order the multipoint MLPG5 formulation (max = $3.91e{-5}$)

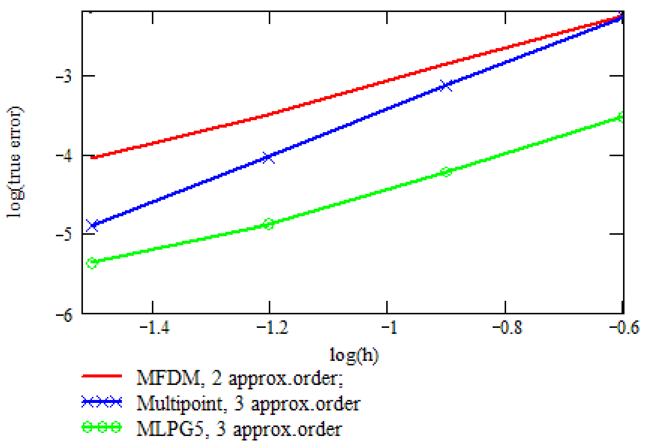

Figure 10. The solution (Prandtl stress function) convergence rate for series of regular meshes

The analysis done required development of appropriate computer program. It was written in C++ Visual Studio using the original advanced visualization tool [6] based on the OpenGL for presentation the meshless method results.

5 FINAL REMARKS

The higher order multipoint meshless finite difference method based on arbitrary cloud of irregularly distributed nodes, moving weighted least squares approximation and the global, local or global-local formulations of boundary value problems, was considered. The paper is focused on a posteriori estimation of the global or local solution and residual errors based on the multipoint MFDM.

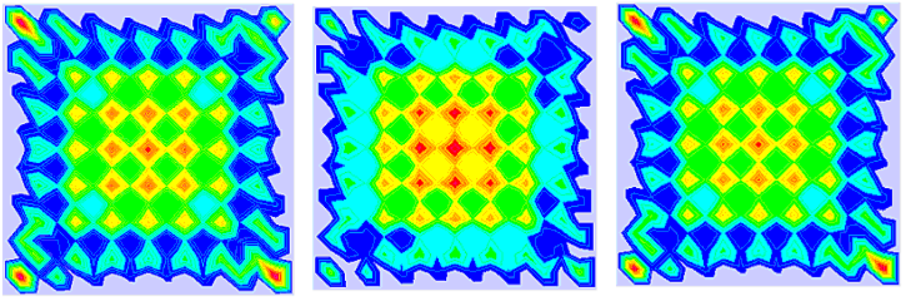

Figure 11. Multipoint error estimation: true solution MFDM error (max $= 8.84e{-}4$); multipoint estimation, local formulation (max $= 7.47e{-}4$); multipoint estimation, MLPG5 (max $= 8.77e{-}4$)

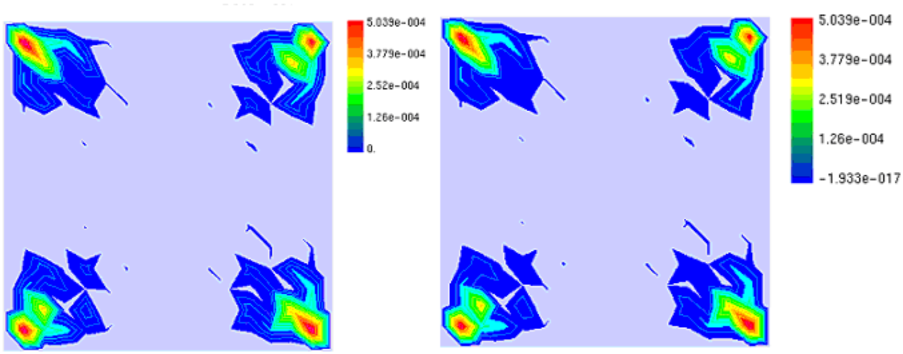

Figure 12. True multipoint 3rd order error (local formulation) and its estimation by multipoint MLPG5

Due to high quality of its results, the multipoint method may be used to develop the reference solutions needed for a posteriori error estimation. A posteriori error analysis of the multipoint MFDM may be applied for two purposes: evaluation of the accuracy of the computed numerical solution and generation of a series of adaptive meshes based on it. The multipoint MFDM based error estimate is, in general, very accurate and quite efficient. The method provides at once higher order solutions and therefore it does not need time consuming remeshing. Multiple preliminary tests confirm high quality of a posteriori error estimation based on the multipoint MFDM.

Further development of application of the multipoint meshless FDM to a priori and a posteriori error analysis, as well as its use to analysis of a large class of b.v. problems is planned.

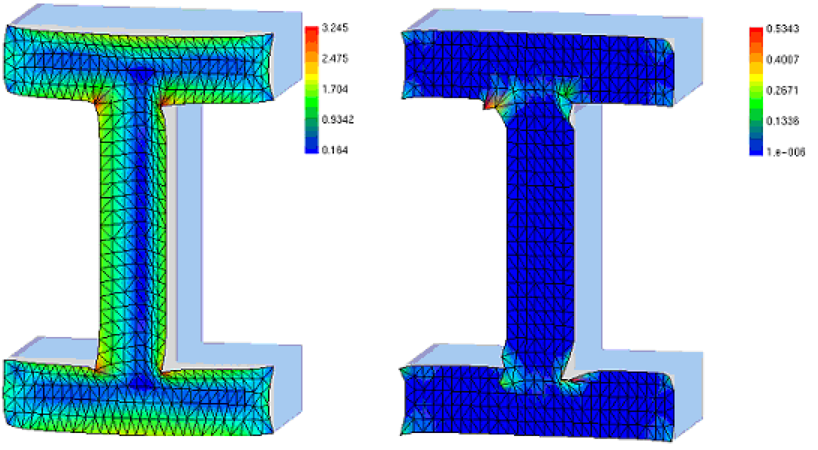

Figure 13. Total shear stress in I-beam and its estimated error

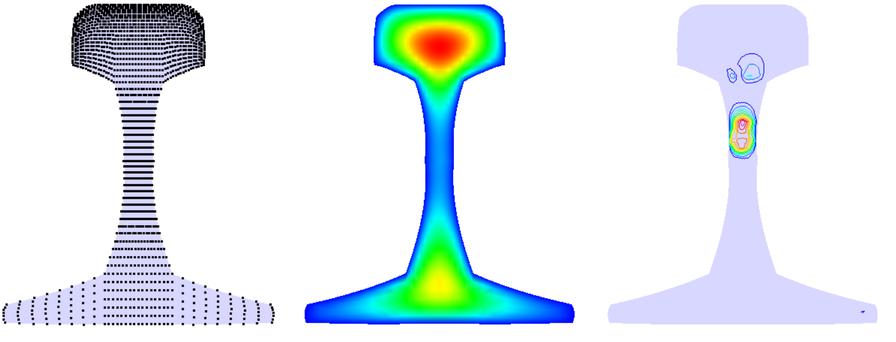

Figure 14. Irregular mesh, Prandtl stress function and estimated error in a railroad rail shape bar

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