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PARAMETER SELECTION AND UNCERTAINTY MEASUREMENT FOR VARIABLE PRECISION PROBABILISTIC ROUGH SET

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Abstract. In this paper, we consider the problem of parameter selection and uncertainty measurement for a variable precision probabilistic rough set. Firstly, within the framework of the variable precision probabilistic rough set model, the relative discernibility of a variable precision rough set in probabilistic approximation space is discussed, and the conditions that make precision parameters α discernible in a variable precision probabilistic rough set are put forward. Concurrently, we consider the lack of predictability of precision parameters in a variable precision probabilistic rough set, and we propose a systematic threshold selection method based on relative discernibility of sets, using the concept of relative discernibility in probabilistic approximation space. Furthermore, a numerical example is applied to test the validity of the proposed method in this paper. Secondly, we discuss the problem of uncertainty measurement for the variable precision probabilistic rough set. The concept of classical fuzzy entropy is introduced into probabilistic approximation space, and the uncertain information that comes from approximation space and the approximated objects is fully considered. Then, an axiomatic approach is established for uncertainty measurement in a variable precision probabilistic rough set, and several related interesting properties are also discussed. Thirdly, we study the attribute reduction for the variable precision probabilistic rough set. The definition of reduction and its characteristic theorems are given for the variable precision probabilistic rough set. The main contribution of this paper is twofold. One is to propose a method of parameter selection for a variable precision probabilistic rough set. Another is to present a new approach to measurement uncertainty and the method of attribute reduction for a variable precision probabilistic rough set.

Keywords: Rough set, probabilistic approximation space, relative discernibility, variable precision probabilistic rough set, approximation reduction

1 INTRODUCTION

Rough set [1, 2] theory is a mathematical theory that has been used since the 1980s to handle uncertain, imprecise, and incomplete information. In recent years, rough set theory has been applied successfully to many fields in computer science and management science, e.g., intelligent data processing [3], data mining [4], big data processing [5], pattern identification [6], image processing [3, 7], decision-making support and process control [3, 8]. Against a varied background of management science situations, several extensions of the Pawlak rough set model have been developed, such as a variable precision rough set model [9], rough set model based on tolerance relations [10, 11], Bayesian rough set model [12], fuzzy rough set model [13], rough fuzzy set model [14], probabilistic rough set model [15], etc. The basic model of classical Pawlak rough set is based on equivalence relations (reflexive, symmetric, and transitive are satisfied), and is represented by a precise inclusion relationship of sets. In this model, equivalence relationship is the crucial concept, and the universe of discourse is divided into a positive region, negative region and boundary region.

But the strict equivalence relationship leads to a relatively broad boundary region, i.e., an uncertainty region. Therefore, finding methods to minimize the boundary region has become a highly discussed issue in both the theoretical and applied study of rough set theory.

The core issue of rough set theory is classification analysis based on a binary relationship for a given domain. The Pawlak rough set model is by nature a qualitative classification model. It classifies based on an equivalence relationship and inclusion relationship between approximation sets, with no consideration of possible overlap of information between set objects. In view of this, incomplete inclusion relations and subordination relations among sets have to be considered. Regarding this constraint of classical Pawlak rough set theory, an effective approach to reduce the boundary region is to extend a strict inclusion relation between sets, and to introduce a majority inclusion relation between sets. This leads to an important extension of Pawlak rough set, i.e., variable precision rough set [9]. Subsequently, many valuable extensions of Ziarko's variable precision rough set were established. Based on Ziarko's idea, Beynon [16] and Katzberg [17] define the model of variable precision rough sets with asymmetric bounds by introducing two parameters to the lower and upper approximations. Sun and Gong [18] present a new generalized model of Ziarko's variable precision rough set based on binary relations over the universe of discourse. In addition, probabilistic rough set theory that combines classical probability theory and Pawlak rough set theory is another effective model to reduce the boundary domain of a Pawlak rough set. In 1987, Wong and Ziarko [19] introduced probabilistic approximation space to the study of rough set and then presented the concept of probabilistic rough set. Subsequently, Yao et al. [20] proposed a more general probabilistic rough set called decision-theoretic rough set. There another perspective to deal with the degree of overlap of an equivalence class with the set to be approximated was given, and an approach to select the needed parameters in lower and upper approximations was presented. As far as the probabilistic approach to rough set theory, Pawlak and Skowron [21], Pawlak et al. [22] and Wong and Ziarko [19] proposed a method to characterize a rough set by a single membership function. By the definition of a rough membership function, elements in the same equivalence class have the same degree of membership. The rough membership may be interpreted as the probability of any element belonging to a set, given that the element belongs to an equivalence class. This interpretation leads to probabilistic rough set [23]. Compared with a classical Pawlak rough set model, variable rough set models and probabilistic rough set models belong to the category of quantitative models, and they can effectively overcome the weaknesses of the Pawlak rough set model in terms of tolerance mechanism and generalization capacity when processing imprecise, inconsistent, and incomplete information.

Finally, in [24], the authors defined another quantitative model by combining a variable precision rough set model and a probabilistic rough set model into a variable precision probabilistic rough set model. The three above-mentioned quantitative extension models based on classical Pawlak rough set models share one common point: a precision parameter needs to be set in advance in their definition of lower and upper approximations. While introduction of a precision parameter can improve the defects of the classical Pawlak rough set theory model, studies of the three models so far have only discussed the existence, domain of the value, and a semantic explanation (management background) of precision parameters. They lack discussion of how to determine precision parameters, i.e., they did not propose a definition of parameters and the selection method. This has constrained the application of the models. Besides, the parameters used in certain conditions are not necessary in some cases. In this case, the credibility of decision rules may be reduced. In fact, the selection of parameters is of vital importance to the selection of decision rules in real life during the application of a variable precision probability rough set model. In this paper, based on [25], we discuss the relative discernibility between sets in probabilistic approximation space. We put forward a method of threshold selection of precision parameters based on the relative discernibility of sets under the precondition of relative discernibility of the set in probabilistic approximation space and consistency of the quality of approximation classification. This makes it practical to select precise parameters in a variable precision probabilistic rough set model. Uncertainty measurement of concepts (objects) in approximation space is another important part of the study of Pawlak rough set theory. As in the Pawlak rough set model, the roughness and precision of variable precision probabilistic rough set only describe uncertainties that come from the approximation space. In fact, the uncertainty of a rough set in approximation space comes from both the approximation space and the approximated set. In view of this, we fully discuss both factors. The concept of fuzzy entropy is introduced into probabilistic approximation space, and an axiomatic approach is employed to put forward a new method to address measurement uncertainty in a variable precision probabilistic rough set. Finally, we briefly introduce an approximation reduction of an information system based on a variable precision probabilistic rough set model.

The rest of this paper is as follows: Section 2 provides the basic concept of binary relations over the universe and briefly reviews the Pawlak rough set theory, variable precision rough set and probabilistic rough set. In Section 3, the discernibility of probabilistic approximation space of variable precision probabilistic rough set is discussed first, then a parameter selection method for a variable precision probabilistic rough set is proposed on this basis. Section 4 investigates the uncertainty measurement of a variable precision probabilistic rough set by introducing the concept of classical fuzzy entropy into probabilistic approximation space. Section 5 discusses the attribute reduction of probabilistic approximation space based on variable precision probabilistic rough set and presents several interesting conclusions. At last, we conclude our research and set out further research directions in Section 6.

2 PRELIMINARIES

In this section, we briefly review the concept of binary relations over a universe as well as the Pawlak rough set model over the universe. Also, we will present the definitions of the variable precision rough set model and probabilistic rough set model.

2.1 Pawlak Rough Set

First of all, we present the definition of an equivalence relation in the universe of discourse.

Definition 1 ([1, 2])**.** Let U be a non-empty and finite universe. Denote $U \times U = \{(x_i, x_j) \mid x_i, x_j \in U\}$. Then the subset $R \subseteq U \times U$ is called an equivalence relation on universe U, if R satisfies the following conditions:

1. Reflexivity: $(x_i, x_i) \in R$, $\forall x_i \in U$;
2. Symmetry: $(x_i, x_j) \in R, \Rightarrow (x_j, x_i) \in R, \forall x_i, x_j \in U$;
3. Transitivity: $(x_i, x_j) \in R, (x_i, x_k) \in R \Rightarrow (x_i, x_k) \in R, \forall x_i, x_j, x_k \in U$.

Let U/R be a set consisting of all equivalent classes based on equivalence relation R in the universe, and let $[x]_R$ represent R equivalent classes that include element $x \in U$. Then $K = (U, R)$ is called a knowledge base or a relationship system, where R represents a cluster of equivalence relationships in domain U.

When there is no risk of confusion, we make no distinction as to equivalence relationship cluster R and equivalence relationship R, i.e., $K = (U, R)$. Meanwhile (U, R) is called Pawlak approximation space [26, 27].

Let (U, R) be the approximation space. For any $X \subseteq U$, we define

$$\underline{R}(X) = \cup\{[x]_R \in U/R \mid [x]_R \subseteq X, x \in U\},$$
$$\overline{R}(X) = \cup\{[x]_R \in U/R \mid [x]_R \cap X \neq \emptyset, x \in U\},$$

the lower approximation and upper approximation of R, respectively.

The lower approximation and upper approximation can also be represented as follows:

$$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\},$$
$$\overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}.$$

$Bn_R(X) = \overline{R}(X) - \underline{R}(X)$ is called the boundary region of X. $Pos_R(X) = \underline{R}(X)$ is the positive region of X, and $Neg_R(X) = U - \overline{R}(X)$ is the negative region of X.

Apparently, $\overline{R}(X) = Pos_R(X) \cup Bn_R(X)$.

Based on the above definition, the following conclusion is obviously valid.

Theorem 1 ([27, 29])**.** Define (U, R) as an approximation space. For any $X \subseteq U$, there is:

1. X is a definable set of R when $\overline{R}(X) = \underline{R}(X)$.
2. X is a rough set of R when $\overline{R}(X) \neq \underline{R}(X)$.

The lower approximation $\underline{R}(X)$ is the union of all elementary sets that are the subsets of X, and the upper approximation $\overline{R}(X)$ is the union of all elementary sets that have a non-empty intersection with X.

The lower (upper) approximation $\underline{R}(X)(\overline{R}(X))$ is interpreted as the collection of those elements of U that definitely (possibly) belong to X.

2.2 Variable Precision Rough Set

The Pawlak rough set model is often too strict when including objects into the approximation regions and may require additional information. A lack of consideration for the degree of overlap between an equivalence class and the set to be approximated unnecessarily limits the applications of Pawlak rough set and has motivated a good deal of new generalization of the Pawlak rough set model. In 1993, Ziarko [9, 28] proposed the variable precision rough set model by introducing the majority inclusion relation over the universe of discourse. In the following, we present Ziarko's variable precision rough set model.

Let U be the universe of discourse. For any two subsets $X, Y \subseteq U$, we define

$$mc(X,Y) = \begin{cases} 1 - |X \cap Y|/|X|, & |X| > 0; \\ 0, & |X| = 0. \end{cases}$$

We call $mc(X,Y)$ the relative error classification rate of set X in relation to set Y.

Let (U,R) be the Pawlak approximation space, for any $X \subseteq U$. We define the β lower approximation and upper approximation of X with respect to approximation space (U,R) respectively as follows:

$$\underline{R}_\beta(X) = \{x \in U \mid mc([x]_R, X) \leq \beta\},$$
$$\overline{R}_\beta(X) = \{x \in U \mid mc([x]_R, X) < 1 - \beta\}.$$

Furthermore, the β positive region, boundary region and negative region of X with respect to approximation space (U,R) can be respectively defined as follows:

$$Pos_\beta(X) = \underline{R}_\beta(X) = \{x \in U \mid mc([x]_R, X) \leq \beta\},$$
$$Bn_\beta(X) = \{x \in U \mid \beta < mc([x]_R, X) < 1 - \beta\},$$
$$Neg_\beta(X) = \{x \in U \mid mc([x]_R, X) \geq 1 - \beta\}.$$

Remark 1. If $\beta = 0$, then the following relation holds:

$$\underline{R}_\beta(X) = \{x \in U \mid mc([x]_R, X) \geq 1\} = \{x \in U \mid [x]_R \subseteq X\} = \underline{R}(X),$$
$$\overline{R}_\beta(X) = \{x \in U \mid mc([x]_R, X) > 0\} = \{x \in U \mid [x]_R \cap X \neq \emptyset\} = \overline{R}(X).$$

This is the Pawlak rough set model.

2.3 Variable Precision Probabilistic Rough Set

In this subsection, we introduce another generalized form of the Pawlak rough set model: variable precision probabilistic rough set.

We first give the concept of probabilistic measurement on the universe of discourse [29, 30].

Definition 2 ([30])**.** Let U be a non-empty finite universe of discourse. The set function $P : 2^U \to [0,1]$ is called the probabilistic measurement on universe U, and satisfies the following conditions:

1. $P(\emptyset) = 0$,
2. $P(U) = 1$,
3. $P(\bigcup_n A_n) = \sum_n P(A_n)$, $A_n \in 2^U, n = 1, 2, \ldots$ and A_n piecewise disjoint.

Let P be the probabilistic measurement on U, $\forall A, B \in 2^U$ and $P(B) > 0$. Then,

$$P(A|B) = \frac{P(A \cap B)}{P(B)},$$

is the conditional probability of occurrence of event A given event B.

Definition 3 ([24])**.** Let U be a non-empty and finite universe of discourse. R is an equivalence relation on U. U/R are equivalence classes formed by R. P is the probabilistic measurement defined on the σ-algebra of measurable subsets of U. Then we call this the probabilistic approximation space.

In the following, we present the definition of variable precision probabilistic rough set with respect to probabilistic approximation space.

Let $A_P = (U, R, P)$ be a probabilistic approximation space. For any $0.5 < \alpha \leq 1$, $X \subseteq U$, the lower approximation $\underline{P}_\alpha(X)$ and upper approximation $\overline{P}_\alpha(X)$ of X with precision parameter α about probabilistic approximation space A_P are, respectively, as follows:

$$\underline{P}_\alpha(X) = \{x \in U \mid P(X|[X]_R) \geq \alpha\},$$
$$\overline{P}_\alpha(X) = \{x \in U \mid P(X|[X]_R) > 1 - \alpha\}.$$

Similarly, the positive region, boundary region and negative region of X about probabilistic approximation space A_P are, respectively, defined as follows:

$$Pos(X,\alpha) = \underline{P}_\alpha(X) = \{x \in U \mid P(X|[X]_R) \geq \alpha\},$$
$$Bn_\alpha(X) = \{x \in U \mid 1-\alpha < P(X|[X]_R) < \alpha\},$$
$$Neg_\alpha(X) = U \backslash \overline{P}_\alpha(X) = \{x \in U \mid P(X|[X]_R) < \alpha\}.$$

3 THE THRESHOLD SELECTION OF VARIABLE PRECISION PROBABILISTIC ROUGH SET

According to the definition of the variable precision probabilistic rough set model, it is well known that roughness of any non-empty subset X ($X \subseteq U$) in approximation space is caused by the existence of boundary region $Bn_\alpha(X)$. Therefore, the boundary region $Bn_\alpha(X)$ affects the discernibility of X, and the boundary region of X varies with parameter α, which further influences the discernibility of the set itself. Because the discernibility of the boundary region of X is relative, a higher discernibility of X in a given classification probability value can be reached if a greater classification probability exists. Based on the above analysis, the following definitions are given.

As is well known, the variable precision probabilistic rough set model [24] is an extension of the existing results. The variable precision probabilistic rough set model was defined by introducing the classical probability measure into the Pawlak approximation space, and then we use the conditional probability of any objects (i.e., the equivalence classes of an element on universe of discourse) with respect to the considered event (i.e., the approximated object set X) instead of the majority include relation used in the original Ziarko's model [9]. Based on the conditional probability, the lower and upper approximations of variable probabilistic rough set model were constructed. That is, the variable precision probabilistic rough set model can be regarded as a probabilistic description of the original Ziarko's model [9] under the framework of probabilistic approximation space, i.e., the variable precision probabilistic rough set model will be degenerated into the original Ziarko's model when we define $P(X|[X]_R) = 1 - \frac{|X \cup [X]_R|}{|[X]_R|} = mc(X,[X]_R)$ (where $|\bullet|$ denotes the cardinality of any set). Therefore, the following conclusions and other results of the variable precision probabilistic rough set model are similar to the results given in Ziarko [9].

Definition 4 ([9])**.** Let (U,R,P) be a probabilistic approximation space. If the α boundary region of X ($X \subseteq U$) about (U,R,P) satisfies $Bn_\alpha(X) = \emptyset$ or, equivalently, $\underline{P}_\alpha(X) = \overline{P}_\alpha(X)$. Then X is called α discernible. Otherwise, X is called α indiscernible.

It is easy to know that the discernibility of X depends on the value of precision parameter α from this definition.

Based on the above definition, the following conclusion can be reached.

Theorem 2 ([9])**.** Let (U, R, P) be a probabilistic approximation space. For any X $(X \subseteq U)$, if X is discernable for parameter α $(0.5 < \alpha \leq 1)$, then X is also discernible for any $\alpha_1 < \alpha$ $(0.5 < \alpha_1 \leq 1)$.

Proof. For $Bn_\alpha(X) = \{x \in U | 1 - \alpha < P(X|[X]_R) < \alpha\}$, if X is discernible on parameter α $(0.5 < \alpha \leq 1)$, then there is $Bn_\alpha(X) = \emptyset$. For any α that satisfies $\alpha_1 < \alpha$ $(0.5 < \alpha_1 \leq 1)$, there is $\{1-\alpha_1 < P(X|[X]_R) < \alpha_1\} \subseteq \{1-\alpha < P(X|[X]_R) < \alpha\}$. This is $Bn_{\alpha_1}(X) \subseteq Bn_\alpha(X)$. So, $Bn_\alpha(X) = \emptyset$. Therefore, X is discernible for any $\alpha_1(\alpha_1 < \alpha)$.

This completes the proof. □

Corollary 1 ([9])**.** Let (U, R, P) be a probabilistic approximation space. For any X $(X \subseteq U)$, if X is indiscernible with respect to parameter α $(0.5 < \alpha \leq 1)$, then X is indiscernible for any $\alpha < \alpha_2(0.5 < \alpha_2 \leq 1)$.

The proof is same as the proof of Theorem 2.

Theorem 2 and Corollary 1 show that the discernibility of any set X increases with the decreasing of the value of precision parameter α. Otherwise, the discernibility of any set X decreases with the increasing value of precision parameter α. That is, for a probabilistic rough set X, there could be a more highly discernible X if a smaller classification precision parameter α was given.

Definition 5. Let U be a non-empty finite universe, and (U, R, P) be a probabilistic approximation space. For any X $(X \subseteq U)$, if $\alpha = 0.5$, we define the absolute boundary region of X about probabilistic approximation space (U, R, P) as:

$$Bn_{0.5}(X) = \{x \in U | P(X|[X]_R) = 0.5\}.$$

Definition 6. Let (U, R, P) be a probabilistic approximation space. For any X $(X \subseteq U)$, if X is indiscernible for any α $(0.5 < \alpha \leq 1)$, then we call X absolutely indiscernible (or absolutely rough set). Otherwise, we call X relatively rough (or weakly discernible).

Theorem 3 ([9])**.** Let (U, R, P) be a probabilistic approximation space. For any X $(X \subseteq U)$, if $\underline{P}_{0.5}(X) \neq \overline{P}_{0.5}(X)$, then X is indiscernible for any precision parameter α $(0.5 < \alpha \leq 1)$.

Proof. Because $\underline{P}_{0.5}(X) \neq \overline{P}_{0.5}(X)$, i.e. $Bn_{0.5}(X) \neq \emptyset$, and for any α $(0.5 < \alpha \leq 1)$, there is $Bn_{0.5}(X) \subseteq Bn_\alpha(X)$. Therefore, $Bn_{0.5}(X) \neq \emptyset$ according to Theorem 2. From Definition 4, it is known that X is α indiscernible.

This completes the proof. □

Corollary 2. Let (U, R, P) be a probabilistic approximation space. Then any X $(X \subseteq U)$ with a non-empty boundary region on (U, R, P) must be indiscernible.

Corollary 3. Let (U, R, P) be probabilistic approximation space. For any X $(X \subseteq U)$, X is an absolutely rough set if and only if $Bn_{0.5}(X) \neq \emptyset$.

It is easy to see that the converse propositions of Theorem 2 and 3 are still valid for the definition of various precision probabilistic rough sets.

Generally speaking, for any set X $(X \subseteq U)$, the discernibility of X about probabilistic approximation space (U, R, P) depends on the value of precision parameter α. In fact, there is always an α for every relative rough set X that makes X discernible at this threshold value. Then we have the following definition.

Definition 7 ([9])**.** Let U be the non-empty finite universe of discourse, and $R \in U \times U$ an equivalence relation on universe U. Let

$$Ind(R, X) = \{\alpha \mid Bn_\alpha(X) \neq \emptyset, \alpha \in (0.5, 1]\}$$

be the whole set of α values that satisfy X and is indiscernible with respect to probabilistic approximation space (U, R, P).

Furthermore, the maximum value of parameter α that satisfies the condition that X is discernible is called the discernible threshold value, denoted as $\gamma_\alpha(P, X)$.

Definition 8 ([9])**.** Let (U, R, P) be a probabilistic approximation space. For any X $(X \subseteq U)$, the discernible threshold value $\gamma_\alpha(P, X)$ satisfies the following conditions:

1. $\gamma_\alpha(P, X) = \mathit{inf}\ Ind(P, X)$,
2. $\gamma_\alpha(P, X) = \min(n_1, n_2)$

where

$$n_1 = 1 - \max\{P(X|[x])|P(X|[X]_R) < 0.5, x \in U\},$$
$$n_2 = \min\{P(X|[x])|P(X|[X]_R) > 0.5, x \in U\}.$$

For any X $(X \subseteq U)$, if X is relatively discernible, then the empty boundary region (i.e., the discernible threshold value boundary region) of X is as follows:

$$Bn_{\gamma_\alpha}(X) = \{x \in U \mid 1 - \gamma_\alpha(P, X) < P(X|[x]_\alpha) < \gamma_\alpha(P, X)\}.$$

Theorem 4. Let (U, R, P) be a probabilistic approximation space. If there are $Bn_\alpha(X) \neq \emptyset$ for any parameter $\alpha \in (0.5, 1]$ and X $(X \subseteq U)$, then X is indiscernible if and only if

$$Bn_\alpha(X) \neq Bn_{\gamma_\alpha}(X), \alpha \in (0.5, 1].$$

In Theorem 4 we talk about α for which X is discernible, therefore, according to Definition 4 we have $Bn_\alpha(X) = \emptyset$.

Based on the former definitions, the following conclusion is clear.

Theorem 5. Let (U, R, P) be a probabilistic approximation space. For any α $(0.5 < \alpha \leq 1)$, X $(X \subseteq U)$, the domain of the probabilistic value X that makes α discernible is as follows:

$$(0.5, \gamma_\alpha(P, X)].$$

In the following, we present a numerical example to demonstrate the method for precision parameter selection given in this paper.

Example 1. Let $U = \{x_1, x_2, \cdots, x_{20}\}$ and let R be an equivalence relation on universe U. P is the probabilistic measurement defined on the σ-algebra of measurable subsets of universe U. Meanwhile, the elementary classes of elements on U with respect to R are as follows, respectively.

$$E_1 = \{x_1, x_2, x_3, x_4, x_5\}, \qquad E_2 = \{x_6, x_7, x_8\}, \qquad E_3 = \{x_9, x_{10}, x_{11}, x_{12}, x_{19}\},$$

$$E_4 = \{x_{17}, x_{18}, x_{20}\}, \qquad E_5 = \{x_{13}, x_{14}, x_{15}, x_{16}\}.$$

Suppose that

$$X = \{x_3, x_5, x_8, x_{14}, x_{15}, x_{16}, x_{18}, x_{19}\}.$$

Take $P(X|E) = \frac{P(X \cap E)}{P(E)}$. Then we have

$$P(X|E_1) = 0.4, \qquad P(X|E_2) = 0.34, \qquad P(X|E_3) = 0.2,$$

$$P(X|E_4) = 0.33, \qquad P(X|E_5) = 0.75.$$

Based on Definition 7, it is easy to calculate and obtain the following results:

$$\begin{aligned} n_1 &= 1 - \max\{P(X|E_1), P(X|E_2), P(X|E_3)\} \\ &= 1 - \max\{0.4, 0.34, 0.33, 0.2\} \\ &= 1 - 0.4 = 0.6, \\ n_2 &= \min\{P(X|E_5)\} = \min\{0.75\} = 0.75. \end{aligned}$$

So, there is $\gamma_\alpha(P, X) = \min\{n_1, n_2\} = \min\{0.6, 0.75\} = 0.6$.

That is, the maximum threshold value that makes X discernible is $\gamma_\alpha(P, X) = 0.6$. Therefore, the corresponding empty boundary region when X is discernible is as follows:

$$Bn_{\gamma_\alpha}(X) = \{x \in U \mid 0.4 < P(X|[x]_R) < 0.6\}.$$

By using the conclusion of Theorem 4, we know that the domain of the value of precision parameter α that makes X discernible is calculated as follows:

$$(0.5, \gamma_\alpha(P, X)] = (0.5, 0.6].$$

This completes the example.

4 UNCERTAINTY MEASUREMENT OF VARIABLE PRECISION PROBABILISTIC ROUGH SET

In the classical Pawlak rough set theory [1, 2], accuracy and roughness are used to characterize the uncertainty of a set and approximation accuracy is employed to depict the accuracy of a rough classification. Pawlak [1, 2] developed the uncertainty measurement of an ordinary set in the universe of discourse. Subsequently, Banerjee [34] studied the uncertainty measurement of a fuzzy set with respect to approximation space. Although these measures are effective, several limitations have been pointed out by many scholars when applying them to certain situations. Therefore, several improved methods of uncertainty measurement for various generalized rough set models (or generalized information systems) have been established in recent years [37].

As is well known, the roughness of any object set with respect to the probabilistic approximation space is induced by the non-empty boundary region, from the definition of the variable precision probabilistic rough set model. There could be a fuzzy membership relation between any object set and the elements in the universe of discourse. Moreover, the fuzzy membership degree between any object set and the elements is determined by the probability $P(X|[x])$.

For any α $(0.5 < \alpha \leq 1)$, $X \subseteq U$, we denote the fuzzy set generated by the conditional probability as $\tilde{X}_P^\alpha$. So, its membership function is defined as follows:

$$\tilde{X}_P^\alpha(x) = P(X|[X]_R) = P(X \cap [X]_R)/P([X]_R), x \in U.$$

In particular, if $P([X]_R) = 0$ or $[X]_R = \emptyset$, then we use the convention that $\tilde{X}_P^\alpha(x) = 1$.

Based on the definition of rough membership function $\tilde{X}_P^\alpha(x)$ on probabilistic approximation space (U, R, P), the lower and upper approximations of variable precision probability rough set by rough membership function are represented respectively as follows:

$$\underline{P}_\alpha(X) = \left\{x \in U \mid \tilde{X}_P^\alpha(x) \geq \alpha\right\},$$

$$\overline{P}_\alpha(X) = \left\{x \in U \mid \tilde{X}_P^\alpha(x) > 1 - \alpha\right\}.$$

That is, the lower and upper approximations of variable precision probabilistic rough set are α cut set and strong $1 - \alpha$ cut set of fuzzy set $\tilde{X}_P$, respectively.

The boundary region and negative region of X are similarly described as follows:

$$Bn_\alpha(X) = \left\{x \in U \mid 1 - \alpha < \tilde{X}_P^\alpha(x) < \alpha\right\},$$

$$Neg_\alpha(X) = U \backslash \overline{P}_\alpha(X) = \left\{x \in U \mid \tilde{X}_P^\alpha(x) \leq 1 - \alpha\right\}.$$

In this section, we will present an approach to uncertainty measurement for variable precision probabilistic rough set by using the concept of fuzzy entropy. The concept of entropy, originally developed by Shannon [33] for communication theory, has been a useful mechanism for characterizing the information in various models and applications in diverse fields. By using Shannon entropy, several conclusions can be established about the uncertainty measurement and knowledge granularity of the rough set in the Pawlak approximation space [35]. As discussed above, there is a fuzzy set generated by the conditional probability of the universe of discourse for any target set X ($X \subseteq U$). So, we use the concept of fuzzy entropy to discuss the uncertainty measurement for a variable precision probabilistic rough set.

Here, we first give the definition of fuzzy entropy as follows.

Let U be a non-empty and finite universe of discourse. Denote as $F(U)$ all the fuzzy subsets of universe U.

Definition 9 ([31, 32, 33])**.** Let mapping $E : F(U) \to [0,1]$. If the following conditions are satisfied:

1. $E(A) = 0$ if and only if A is a crisp set on U;
2. $E(A) = 1$ if and only if $\mu_A(x) = 0.5, \forall x \in U, A \in F(U)$;
3. If $D(A, 0.5) \geq D(B, 0.5)$, then $E(A) \leq E(B), \forall A, B \in F(U)$;
4. $E(A) = E(A^c)$ (A^c is the complementary set of A)

where $D(A,B) = \sqrt{\frac{1}{|U|}\sum_{x\in U}(\mu_A(x) - \mu_B(x))^2}$ indicates the distance between two rough sets.

Then E is called an entropy on $F(U)$.

With the axiomatic definition of fuzzy entropy, a roughness measurement of a variable precision probabilistic rough set is put forward.

Definition 10. Let (U, R, P) be a probabilistic approximation space. For any $X \in U$, the roughness measurement of rough set X with respect to (U, R, P) is defined as follows:

$$f(\tilde{X}_P^\alpha) = \frac{4}{|U|}\sum_{x\in U}\tilde{X}_P^\alpha(x)\left(1 - \tilde{X}_P^\alpha(x)\right).$$

Lemma 1. Let (U, R, P) be probabilistic approximation space. For any X ($X \subseteq U$), $f\left(\tilde{X}_P^\alpha\right)$ is a fuzzy entropy over $F(U)$.

Proof. For any X ($X \subseteq U$), it is easy to verify that the relation $0 \leq \tilde{X}_P^\alpha(x)\left(1 - \tilde{X}_P^\alpha(x)\right) \leq \frac{1}{4}$ holds. Therefore, $0 \leq f\left(\tilde{X}_P^\alpha\right) \leq 1$.

In the following, we will verify the conditions given in Definition 9 one by one for the roughness measurement $f\left(\tilde{X}_P^\alpha\right)$ of a variable precision probabilistic rough set.

1. If $\tilde{X}_P^\alpha$ is a crisp set, then for any X there is $\tilde{X}_P^\alpha(x) = 0$ or $\tilde{X}_P^\alpha(x) = 1$. So, $f\left(\tilde{X}_P^\alpha\right) = 0$. On the other hand, if $f\left(\tilde{X}_P^\alpha\right) = 0$, then there is $\tilde{X}_P^\alpha(x)\left(1 - \tilde{X}_P^\alpha(x)\right) = 0$ for any $x \in U$. Thus, there is $\tilde{X}_P^\alpha(x) = 0$ or $\tilde{X}_P^\alpha(x) = 1$, i.e., $\tilde{X}_P^\alpha(x)$ is a crisp set.
2. If $x \in U$, $\tilde{X}_P^\alpha(x) = 0.5$, then there is $1 - \tilde{X}_P^\alpha(x) = 0.5$. Moreover, there is $\tilde{X}_P^\alpha(x)\left(1 - \tilde{X}_P^\alpha(x)\right) = 0.25$. So, $f\left(\tilde{X}_P^\alpha\right) = \frac{4}{|U|}\sum_{x\in U}\frac{1}{4} = 1$.

   On the contrary, suppose that $f\left(\tilde{X}_P^\alpha\right) = 1$. Then, there is $\tilde{X}_P^\alpha(x)\left(1 - \tilde{X}_P^\alpha(x)\right) = \frac{1}{4}$, which holds by the above discussion. This proves that $\tilde{X}_P^\alpha(x) = 0.5$. In other words, $\tilde{X}_P^\alpha(x)$ arrives at the maximum fuzziness.
3. If $D\left(\tilde{X}_P^\alpha, 0.5\right) \geq D\left(\tilde{Y}_P^\alpha, 0.5\right)$, for any $X, Y \subseteq U$, there is

$$\begin{aligned}
f\left(\tilde{X}_P^\alpha\right) &= \frac{4}{|U|}\sum_{x\in U}\tilde{X}_P^\alpha(x)\left(1 - \tilde{X}_P^\alpha(x)\right) \\
&= \frac{4}{|U|}\sum_{x\in U}\left(0.5 + \tilde{X}_P^\alpha(x) - 0.5\right)\left(0.5 - \left(\tilde{X}_P^\alpha(x) - 0.5\right)\right) \\
&= \frac{4}{|U|}\sum_{x\in U}\left(0.25 - \left(\tilde{X}_P^\alpha(x) - 0.5\right)^2\right) = 1 - \frac{4}{|U|}\sum_{x\in U}\left(\tilde{X}_P^\alpha(x) - 0.5\right)^2 \\
&\leq 1 - \frac{4}{|U|}\sum_{x\in U}\left(\tilde{Y}_P^\alpha(x) - 0.5\right)^2 = f\left(\tilde{Y}_P^\alpha\right).
\end{aligned}$$

4. For any $x \in U$, there is $\left(\tilde{X}^c\right)_P^\alpha(x) = 1 - \tilde{X}_P^\alpha(x)$, $\left(\tilde{X}^c\right)_P^\alpha(x) = 1 - \tilde{X}_P^\alpha(x) = \left(1 - \tilde{X}_P^\alpha(x)\right)\tilde{X}_P^\alpha(x)$. That is, there is $f\left(\tilde{X}_P^\alpha\right) = f\left(\left(\tilde{X}^c\right)_P^\alpha\right)$.

Therefore, according to the results of 1., 2., 3. and 4., we know that $f\left(\tilde{X}_P^\alpha\right)$ is a fuzzy entropy on $F(U)$.

This completes the proof. □

By using Definition 9 and Lemma 1, the following results about the uncertainty measurement of variable precision probabilistic rough set are clear.

Theorem 6. Let (U, R, P) be a probabilistic approximation space. For any $0.5 < \alpha \leq 1$ and $X \subseteq U$, there is:

1. $f(U) = f(\emptyset) = 0$.
2. For any $X \in U$, if $\underline{P}_\alpha(X) = \overline{P}_\alpha(X)$, there is $f\left(\tilde{X}_P^\alpha\right) = 0$.
3. For any $X \in U$, if $[x] \neq \emptyset$, there is $f\left(\tilde{X}_P^\alpha\right) = f\left(\left(\tilde{X}^c\right)_P^\alpha\right)$.

Proof. It can be verified directly by the definitions. □

5 ATTRIBUTE REDUCTION OF VARIABLE PRECISION PROBABILISTIC ROUGH SET

In general, objects are described by different attributes. However, it is not necessary to know all attributes for the classification of information systems. That is, some attributes are unnecessary and do not affect the result of classification when removed from the attribute set. Meanwhile, some attributes are indispensable to the result of classification and affect the result when removed from the attribute set. Furthermore, some attributes are relatively necessary for the classification and may determine the result by associating with other attributes. The attribute reduction presents a minimum attribute subset completely describing the classification as the original attribute set for information systems [26, 36, 37, 38, 39, 40, 41]. This subsection will investigate the problem of attribute reduction for an information system based on variable precision probabilistic rough set.

Let U be a non-empty finite universe of discourse. $\Re$ is a family of equivalence relationships over the universe U. Let $K \subseteq \Re$ ($K \neq \emptyset$) and the intersection of all equivalence relations in K be called indiscernible relations on [3]. We denote the intersection of all equivalence relations as $ind(K)$.

Definition 11. Let $S = (U, V, A, F)$ be an information system. $C, D \subseteq A$ are respectively a conditional attribute and decision attribute of probabilistic approximation space (U, R, P). Then, the α ($0.5 < \alpha \leq 1$) approximation dependence of conditional attribute C and decision attribute D of probabilistic approximation space is defined as follows:

$$d_\alpha(C, D) = \frac{|\bigcup_{P(X|[X]_R) \geq \alpha}\{X \mid X \in U/ind(D)\}|}{|U|}.$$

By the definition of approximation dependence in Definition 11, we can easily know that this concept is a natural generalization of the classical approximation dependence in Pawlak rough set theory.

Specifically, $d_\alpha(C, D)$ will degenerate into the classical approximation dependence in Pawlak rough set theory when $\alpha = 1$.

Here, the attribute reduction means that the minimum attribute subset of the conditional attributes results in the same approximation dependence with respect to the decision attribute. We then present the definition of the α approximation attribute reduction for an information system based on variable precision probabilistic rough set theory by using the concept of α ($0.5 < \alpha \leq 1$) approximation dependence as follows.

Definition 12. Let $S = (U, V, A, F)$ be an information system. $C, D \subseteq A$ are respectively conditional attribute and decision attribute of probabilistic approximation space (U, R, P). Then the α approximation reduction $Red_\alpha(C, D)$ is a minimum attribute subset of conditional attribute set C and satisfies the following conditions:

1. $d_\alpha(C, D) = d_\alpha(Red_\alpha(C, D), D)$;
2. The equation given in (1) will no longer be valid when any one attribute is removed from $Red_\alpha(C, D)$.

Theorem 7. Let $S = (U, V, A, F)$ be an information system. Then the approximation reduction of approximation space S always exists for any precision parameter α $(0.5 < \alpha \leq 1)$.

Proof. If for any $c \in C \subseteq A$, and satisfies $R_{C-\{c\}} \neq R_C$, then C is a reduction of information system $S = (U, V, A, F)$. Otherwise, for any $c \in C \subseteq A$, and satisfies $R_{C-\{c\}} = R_C$ hold. Then, we consider the new attribute subset $C_1 = C - \{c\}$. Meanwhile, if for any $c_1 \in C_1 \subseteq A$, there is $R_{C_1-\{c_1\}} \neq R_C$ hold, then, C_1 is a reduction of information system $S = (U, V, A, F)$. Otherwise, for any $c_1 \in C_1 \subseteq A$, there is $R_{C_1-\{c_1\}} = R_C$ hold. Next, we further consider $C_2 = C_1 - \{c_1\}$ and repeat the above process. So, we will find the minimum attribute subset $C^* \subseteq C$ that satisfies the relationships $R_{C^*} = R_C$ and $R_{C^*-\{c\}} \neq R_C$ for any $c \in C^*$. Therefore, C^* is the reduction of information system $S = (U, V, A, F)$.

This completes the proof. □

In general, there may not be only one reduction for information system $S = (U, V, A, F)$ because there may be different combinations among the elements of the attribute set. In practice, we focus on finding only one of the reductions for the information system $S = (U, V, A, F)$.

6 CONCLUSIONS

By introducing precision parameter α $(0.5 < \alpha \leq 1)$ into classical probabilistic rough set, the variable precision probabilistic rough set converts two parameters in the upper (lower) approximation of probabilistic rough set into one parameter. This further improves the robustness and adaptability of the model, extends classical Pawlak rough set theory, and allows rough set theory to process random, uncertain, and inconsistent data information more effectively. Meanwhile, new models and approaches are proposed in which rough set theory is applied to solve decision-making problems with uncertainty in actual situations in management science.

In this paper, we discuss two issues for the variable precision probabilistic rough set model: the relative discernibility of any object set in the universe and the uncertainty measurement of the variable precision probabilistic rough set. For the first aspect content, it is well known that the variable precision probabilistic rough set model [24] is an extension of the existing results by combining the variable precision rough set model [9] and the probabilistic rough set model [22, 23]. The basic idea of the variable precision probabilistic rough set model was defined by introducing the classical probability measure into the Pawlak approximation space, and the conditional probability of any objects (i.e., the equivalence classes of an element on universe of discourse) with respect to the considered event (i.e., the approximated object set X) instead of the majority include relation used in the original Ziarko's model [9]. However, the variable precision probabilistic rough set model will be degenerated into the original Ziarko's model when we define $P(X|[X]_R) = 1 - \frac{|X \cup [X]_R|}{|[X]_R|} = mc(X, [X]_R)$. At the same time, the variable precision probabilistic rough set model will be degenerated into the probabilistic rough set model when we define $1-\alpha = \beta$ in the upper approximation. So, all the results about the relative discernibility of any object set of universe with the variable precision probabilistic rough set are the generalization of the original Ziarko's model [9]. Similarly, the results will be degenerated into the corresponded conclusions of Ziarko's model when we define $P(X|[X]_R) = 1 - \frac{|X \cup [X]_R|}{|[X]_R|} = mc(X, [X]_R)$. Therefore, the results given in Ziarko's model [9] are based on the classical Pawlak approximation space and the results obtained in this paper are under the framework of probabilistic approximation space. For the second aspect, we investigate the uncertainty measurement of any object set with respect to the variable precision probabilistic rough set model. By introducing the concept of the fuzzy entropy into the probabilistic approximation space, we establish a new approach to measure the uncertainty of the approximation quality of any object set with respect to the variable precision probabilistic rough set model. Further, we explore the attribute reduction for the information systems based on the variable precision probabilistic rough set model. Factually, random information acquisition and decision-making problems under general relations or one kind of certain relation are used more widely, and this offers direction for our further study.

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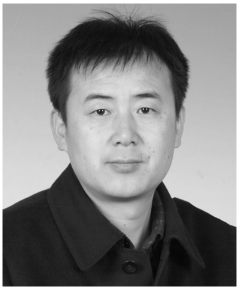

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