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GRANULAR PARTITION AND CONCEPT LATTICE DIVISION BASED ON QUOTIENT SPACE

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Abstract. In this paper, we investigate the relationship between the concept lattice and quotient space by granularity. A new framework of knowledge representation – granular quotient space – is constructed and it demonstrates that concept lattice classing is linked to quotient space. The covering of the formal context is firstly given based on this granule, then the granular concept lattice model and its construction are discussed on the sub-context which is formed by the granular classification set. We analyze knowledge reduction and give the description of granular entropy techniques, including some novel formulas. Lastly, a concept lattice constructing algorithm is proposed based on multi-granular feature selection in quotient space. Examples and experiments show that the algorithm can obtain a minimal reduct and is much more efficient than classical incremental concept formation methods.

Keywords: Classification set, granule, quotient space, concept lattice, entropy

1 INTRODUCTION TO THE FIELD

FCA (Formal Concept Analysis) is usually called concept lattice. It is a mathematical framework for discovery and design of concept hierarchies from an information system called a formal context [10]. All concepts in a formal context form a concept lattice and can be depicted by the Hasse diagram, where each node expresses a formal concept. The concept lattice is the core structure of data in formal concept analysis, from which a set of objects and a set of attributes are uniquely reflected from each other and the relations between the generalized concept and the specialized concept are described. Formal concept analysis is an important theory for data analysis and knowledge discovery [8, 4, 6, 14, 21, 3]. Many researchers have done some research about the concept lattice and granular computing. Wu [11] introduced attribute granules in formal contexts. The mathematical structure of attribute granules was investigated. Based on the theory of concept lattice and fuzzy concept lattice, Zhang et al. [20] established a mathematical model of a concept granular computing system. And various variable threshold concept lattices and fuzzy concept lattices were then investigated. For this research, concept granules, sufficiency information granules and necessity information granules which were used to express different relations between a set of objects and a set of attributes were proposed. Approaches to construct sufficiency and necessity information granules were also shown. Wu et al. [12] proposed a granular consistent set and a granular reduct in the formal context. Discernibility matrices and Boolean functions were, respectively, employed to determine granular consistent sets and calculate granular reducts in formal contexts. Based on granular computing theory, Zhang et al. [19] proposed a new formal concept analysis method with various granularity levels of attributes. Many theorems and relationships among the concept lattices and its sublattices generated from formal context with various granularity levels of attributes were analyzed in detail. Yao et al. [15] proposed a framework for studying a particular class of set-theoretic approaches to granular computing. They thought that a granule was a subset of a universal set, a granular structure was a family of subsets of the universal set, and the relationship between granules was the standard set-inclusion relation. Qiu et al. [7] introduced extent-intent and intent-extent operators between two complete lattices and established a mathematical model for concept granular computing system. They proved that the set of all concepts in this system was a lattice with the greatest element and the least element. This framework included formal concept lattices from formal contexts, L fuzzy concept lattices from L fuzzy formal contexts and three kinds of variable threshold concept lattices. Other studies for the extension of concept lattice models for granular computing are also underway [5, 9, 17, 22].

2 RELATED WORK

The basic idea of the quotient space mainly describes a class of problems through a triple [16, 18], such as (X, f_a, T), where X represents the universe, f_a represents the set of attributes in the universe, T is the topological structure of X and represents relationships between members in the universe. The solution of the problem can be regarded as discussing the universe X, the attributes and related structures.

Given a special size, i.e. when a special equivalence relation is confirmed, we would get a corresponding quotient set $[X]$. Then we can define the triples $([X], [f_a], [T])$ as the quotient space which relates with R, where $[f_a]$ and $[T]$ correspond respectively to quotient attribute function of quotient set X and quotient structure. If we combine different quotient sets with the corresponding quotient spaces, we will get the world of different sizes for the problem. The purpose of quotient space theory is to study the relationship of the quotient spaces, the decomposition, comprehension and combination of the quotient space, and the reasoning between and within the quotient spaces. In this theoretical system, granulation, computing and processing are the basic problems.

In the concept lattice (definition in Section 4), thanks to $X \subseteq U$, $B^* \subseteq U$, $B \subseteq A$, $X^* \subseteq A$, so X, B^*, B and X^* are part of the whole, they can be regarded as the visual granules of formal context (U, A, I). Therefore, it shows that the concept lattice is closely related to the visual granules X, B, X^*, B^*. This is the important reason when the concept lattice is seen as the granular computing. As the results of the current research, the concept lattice combined with the quotient space has not yet been reported.

3 PAPER OBJECTIVE

Quotient space and concept lattice are generally regarded as important researches on granular computing because they all can be described by abstract granules. Our objective is to provide an approach to organizing information for constructing concept lattice, complementing granular computing-based direct search. Our research will provide a way for formulization on granular computing.

This paper is organized as follows. In Section 4, we study the concept representation mechanisms involved in the lattice model of granular computing. They directly produce the granular quotient space. Section 5 gives the description of granular entropy techniques, including some novel formulas. It can be used to compute the minimal reduct. Section 6 explains the results of empirical study to apply proposed algorithm of constructing concept lattice. Sections 7 describes our future works and draw conclusions.

4 CONCEPT LATTICE AND QUOTIENT SPACE

Now, we first introduce the definition of the formal concept analysis [2].

A formal context is a triple (U, A, I), where $U = \{u_1, u_2, \ldots, u_n\}$. U is called the universe of discourse, a nonempty and finite set of objects. $A = \{a_1, a_2, \ldots, a_m\}$ is a nonempty and finite set of attributes, and $I \subseteq U \times A$ is a binary relation between U and A with $(x, a) \in I$ indicating that the object x owns the attribute a.

For $X \subseteq U$ and $B \subseteq A$, let us define a pair of operators "*": $X^* = \{a \in A | \forall x \in X, (x, a) \in I\}$, $B^* = \{x \in U | \forall a \in B, (x, a) \in I\}$. That is, X^* is the maximal family of the attributes that all the objects in X have in common and B^* is the maximal family of the objects shared by all the attributes in B.

Definition 1 ([2])**.** Let (U, A, I) be a formal context. For $X \subseteq U$ and $B \subseteq A$, the ordered pair (X, B) is called a formal concept (or simply a concept) if it satisfies $X^* = B$ and $B^* = X$. Here, X^* and B^* are termed, respectively, as the extent and the intent of the formal concept (X, B). The set of all formal concepts of (U, A, I) is denoted by $\kappa(U, A, I)$.

The fundamental theorem of FCA states that the set of all formal concepts on a given context with the ordering $(X_1, B_1) \leq (X_2, B_2)$ if and only if $X_1 \subseteq X_2$ is a complete lattice called the concept lattice, in which the infima and suprema are given by $\bigwedge_{i\in J}(X_i, Y_i) = \left(\bigcap_{i\in J} X_i, ((\bigcup_{i\in J} Y_i)^*)^*\right) = (\bigcap_{i\in J} X_i, (\bigcap_{i\in J} X_i)^*))$, $\bigvee_{i\in J}(X_i, Y_i) = \left(((\bigcup_{i\in J} X_i)^*)^*, \bigcap_{i\in J} Y_i\right) = ((\bigcap_{i\in J} X_i)^*), \bigcap_{i\in J} X_i)$.

We have following definition, when the attribute of concept lattice is not binary attributes $(0, 1)$ but the multi-valued attributes.

Definition 2 ([27, 26, 28])**.** A quadruple (U, A, V, f) is called an information system, where:

- U is a non-empty finite set of objects;
- A is a non-empty finite set of attributes;
- $V = \bigcup_{a\in A} V_a$, where V_a is a value-universe of the attribute a;
- $f : U \times A \to V$ is an information function, such that for all $a \in A$ and $x \in U$ it holds that $f(x, a) \in V_a$.

If f is a total function, i.e. $f(x, a)$ is defined for all $x \in U$ and $a \in A$, then the information system is called complete; otherwise it is called incomplete.

We can study the so-called concept knowledge system if the power sets of objects and attributes meet some relations in the information system.

Definition 3 (Concept knowledge system [6])**.** A concept knowledge system can be defined as a four-tuple: (U, A, L, H). Here U is a finite nonempty object set. A is a finite nonempty attribute set. $P(U)$ is the power set of U. $P(A)$ is the power set of A. $X_1, X_2 \subseteq U; B_1, B_2 \subseteq A$.

- $L : P(U) \longrightarrow P(A)$ is called extension-intension operator if L satisfies (L_1) $L(\emptyset) = A$; $L(U) = \emptyset$; $(L_2) L(X_1 \bigcup X_2) = L(X_1) \bigcap L(X_2)$. $L(X)$ is a common attributes set of X.
- $H : P(A) \longrightarrow P(U)$ is called intension-extension operator if H satisfies (H_1) $H(\emptyset) = U$; $H(A) = \emptyset$; $(H_2) H(B_1 \bigcup B_2) = H(B_1) \bigcap H(B_2)$. $H(B)$ is a common objects set of B.

Further, it meets $(LH) : H(L(X)) \supseteq X, L(H(B)) \supseteq B$.

Definition 4 ([6])**.** Let (U, A, L, H) be a concept knowledge system, $X \subseteq U$ and $B \subseteq A$. The pair (X, B) is called a concept, which is known as the granular concept if $X = H(B)$, $B = L(X)$. $\Im(U, A, L, H) = \{(X, B) | H(B) = X, L(X) = B\}$ is the set of all granular concepts.

The concept knowledge system is an extension of general information system. It shows that any object set and any attribute set can be constituted as a concept information granule and even a concept by iteratively computing. This process describes the procedure of human cognition step by step.

Definition 5 ([6])**.** Let (U, A, L, H) be the concept knowledge system, $X \subseteq U$ and $B \subseteq A$, the pair (X, B) is called a concept information granule if $X \subseteq H(B), B \subseteq L(X)$.

Theorem 1 ([6])**.** Let (U, A, L, H) be a concept knowledge system. The knowledge system generated by formal context (U, A, I) is the same as (U, A, L, H).

We can set up a problem space for the objective of our study. The aim of representing a problem at different granularities is to enable the computer to solve the same problem at different granularity hierarchically.

Definition 6 (Quotient space [18])**.** Let (X, f_a, T) be a problem space. Suppose that X represents the universe with the finest granular size. There is a coarse-granular universe denoted by $[X]$ that forms a new problem space $([X], [f_a], [T])$. $([X], [f_a], [T])$ is called a quotient space of (X, f_a, T).

Quotient space theory is a new kind of mathematical tool to deal with problems based on different grain sizes. The problem here is with a triple (X, f, T), where X is a set of objects, known as the universe, f is a function of X for the universe. $f(u)$ said attribute values u related problems when $u \in X$. The function $f(u)$ generally depends on the specific situation. T is made up of a subset of the universe X. It forms the topological structure [31]. For example, a given problem (X, f, T), R is equivalence relation on X. So get the division $[X] = \{X_1, X_2, \ldots, X_n\}$ of R about X, coupled with the topological T, we can get the quotient space $([X], [f], [T])$. The $[f]$ is the function of $[X]$ as the universe. For $X_i \in [X]$, function value $[f](X_i)$ is determined by values that f acts each element in the X, and the determination method to different problem may be different. $[T]$ is the quotient topology. Its generation method is as follows:

- Let $p : X \to [X]$ be a function from X to $[X]$, makes for $x \in X$, if $x \in X_i$, then $p(x) = X_i$. $[T] = \{Y | Y \subseteq [X]$ and $p^{-1}(Y) \in T\}$. Here $p^{-1}(Y)$ said the original image set of Y about p. It can be proved that $[T]$ is the topology on $[X]$ [31], called $[T]$ the quotient topology on $[X]$. Due to the element in the $[X]$ is the equivalence class, so the quotient space $([X], [f], [T])$ is closely related to the equivalence relation. Equivalence class is the part of X, can be seen as intuitive grains of problems (X, f, T). It is natural that $[X]$ is determined by the intuitive grain. So the structure of the quotient space $([X],[f],[T])$ embodies the grain to approximate the problems (X,f,T). Since the different equivalence relation for different quotient space, so one of the most important aspects of the quotient space theory is the relation between quotient spaces, such as all the quotient space corresponding equivalent relations on X, and contain relations between equivalent relation form a lattice [30, 23].

Definition 7. Let G be a visual granule of the universe. If there is space $\langle U, Form(U)\rangle$ and n-formula $(n \geq 1)$, $\varphi \in Form(U)$, such that $G = |\varphi|$, then G can be described in granular space $\langle < U, Form(U)\rangle$.

Theorem 2 (Formal context to quotient space)**.** Let (U,A,L,H) be a concept knowledge system generated by formal context (U,A,I), $X \subseteq U$ and $B \subseteq A$. Then $([U],I,[A])$ constitutes quotient space. Granule $L(X)(\subseteq A)$ and $H(B)(\subseteq U)$ can be described in the quotient space.

Proof. Since $I \subseteq U \times A$, we can obtain $B = L(X) \subseteq A$ for $\forall X \subseteq U$, that is, $X^* = B, B^* = X$. There exists $X^* = B = L(X)$. So, I is f. $([U],I,[A])$ constitutes the quotient space. For $([U],I,[A])$, $[U]$ corresponds to granular space $\langle [U], Form([U])\rangle$, where $Form([U])$ is the set of logical formulas on $[U]$.

If using Q to represent X, then $Q(x,a) \in Form([U])$ for $a \in A, x \in [U]$, and $Q(x,a)$ is an atomic formula on $[U]$. $Q(x,a) = \{x|x \in [U]$ and $\langle x,a\rangle \in Q\} = \{x|x \in [U]$ and $\langle x,a\rangle \in I\}$. Let $B = \{a_{i_1}, a_{i_2}, \ldots, a_{i_r}\}$, $H(B) = \{x|x \in [U]$ and $\langle x,a\rangle \in I$ for $\forall a \in B\} = \{x|x \in [U]$ and $\langle x,a_{i_1}\rangle \in I\} \cap \{x|x \in [U]$ and $\langle x,a_{i_2}\rangle \in I\} \cap \ldots \cap \{x|x \in [U]$ and $\langle x,a_{i_r}\rangle \in I\} = \{x|x \in [U]$ and $\langle x,a_{i_1}\rangle \in Q\} \cap \{x|x \in [U]$ and $\langle x,a_{i_2}\rangle \in Q\} \cap \ldots \cap \{x|x \in [U]$ and $\langle x,a_{i_r}\rangle \in Q\} = Q(x,a_{i_1}) \cap Q(x,a_{i_2}) \cap \ldots \cap Q(x,a_{i_r}) = Q(x,a_{i_1}) \wedge Q(x,a_{i_2}) \wedge \ldots \wedge Q(x,a_{i_r})$. If we let $\varphi(x) = Q(x,a_{i_1}) \wedge Q(x,a_{i_2}) \wedge \ldots \wedge Q(x,a_{i_r}), \varphi(x) \in Form([U])$, $H(B) = \varphi(x)$. Therefore, granular $H(B)(\subseteq U)$ can be described in the quotient space. Similarly, we can prove $L(X)$. □

The attributes set A in (U,A,V,f) can be divided into several disjoint subsets with equivalence relation. Each subset is an equivalence class of an attribute. The equivalence classes that constitute quotient space form a lattice.

Theorem 3. Let (U,A,V,f) be an information system, $P(A)$ be a power set of A. Then a quotient space can be constituted by $B \in P(A)$ on A. And all quotient spaces and containing relations between the equivalence relations form a lattice.

Proof. Note that $B \in P(A)$. Then there exists a partition $U/R = \{[u]_R|u \in U\}$ determined by $B's$ equivalence relation R on U. Denoted by $[X]_B = \{[u]_R|u \in U\}$. Obviously, it forms a quotient space. If another $C \in P(A)$, we can see that $[X]_B \vee [Y]_C = [X]_B \cap [Y]_C, [X]_B \wedge [Y]_C = [X]_B \cup [Y]_C$. It is easy to know that they satisfy the commutative, associative and absorption law. $[\emptyset]_A$ and $[U]_A$ are the least element and the greatest element, respectively, so $(P(A), \wedge, \vee, [\emptyset]_A, [U]_A)$ is bounded lattice. □

Because the quotient space is to consider both the object and the attribute, we study the concept information granule of the quotient space and their characteristics in this section.

For an information system (U, A, V, f), by equivalence relation R of $B \subseteq A$ on U, we can granulate information $U/B = [B]_R = [v_1^B, v_2^B, \ldots, v_k^B]_R$, where v_i^B are the possible values of attributes, and then obtain atomic quotient space $U/[\{v_i^B\}]_R = X_{R_i} \subseteq P(U)$, a finer quotient space.

U	*Time*	*Item*	*Location*
1	Jan	cosmetic	Beijing
2	Feb	foodstuff	Beijing
3	Jan	cosmetic	Shanghai
4	Feb	cosmetic	Shanghai
5	Mar	foodstuff	Wuhan
6	Mar	foodstuff	Wuhan

Table 1. An information system

For example, in Table 1, $U/(time, item) = [time, item]_R = \{\{1, 3\}, \{2\}, \{4\}, \{5, 6\}\}$, taking $v_1^B = \{Jan, cosmetic\}$ for $B = (time, item)$, we obtain that $U/[\{v_i^B\}]_R = U/[\{Jan, cosmetic\}]_R = \{\{1, 3\}\}$.

Theorem 4. An objects set of concept in an information system (U, A, V, f) must be included in quotient space of equivalence relation R. An atomic quotient space corresponds to a concept if it has not sub-quotient space.

Proof. For information system (U, A, V, f), defining the operator by equivalence relation R on $P(A) \to P(U)$ and $P(U) \to P(A)$ can build concept knowledge system (U, A, L, H). (X, B) is the concept if $X \subseteq U$ and $B \subseteq A$ satisfy $L(X) = B$, $H(B) = X$. Obviously, $X \in U/R_B$. Therefore, object set of the concept is contained in the classification of quotient space. For X_B, if there does not exist $X'_{B'} \subset X_B$, it indicates that there is no $B \subset B'$, i.e. $L(X_B) = B, H(B) = X_B$, so (X_B, B) is a formal concept. □

Example 1. The concept lattice of Table 1 is shown in Figure 1. It is easy to see that the equivalence classes of quotient space on equivalence relation R are present in the objects sets of concept lattice.

In fact, the quotient space is a kind of division. From the view of the covering, we have the following definition.

Definition 8. Given a formal context (U, A, I), a relation is defined as follows: $R_s = \{(x, y) \in U \times U : \forall a \in \{x\}^*, yIa\}$. There is a set $R_c(x) = \{y | (x, y) \in R_s, y \in U\}$.

Obviously, $R_c(x) = (\{x\}^*)^*$.

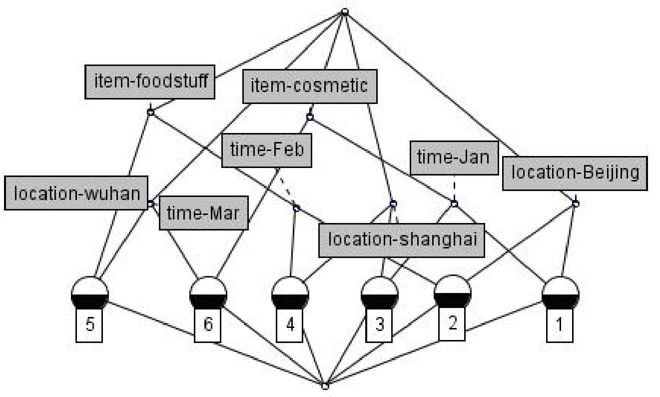

Figure 1. Concept lattice of Table 1

Definition 9. Let U be an object set. $I' = I \cap (U \times B), B \subseteq A$ is a binary relation between U and B that satisfies $\forall x \in U, \forall a \in B, xI'a$. (U, B, I') is a sub-context of (U, A, I).

Theorem 5. Suppose that (U, B, I') is a sub-context of the $(U, A, I), B \subseteq A$. $(R_c(x), (R_c(x))^*), \forall x \in U$ is a formal concept.

Proof. By the definition of $Rc(x)$, $(Rc(x), (Rc(x))^*)$ is a formal concept of (U, A, I). And $(R_c(x)^*_B) = (R_c(x))^*$, $((R_c(x))^*)^*_B = (\{x\}^*)^*_B$, there is $(\{x\}^*)^*_B = (\{x\}^*)^* = R_c(x)$. So $(R_c(x), (R_c(x))^*)$ is a formal concept of the sub-context. □

Theorem 6. Each formal concept on (U, A, I) can be generated by $\bigvee$ operation of the formal concepts that are formed by R_c classified sets. If $\bigcup_{x \in U} R_c(x) = U$, then $\bigcup_{x \in U} (R_c(x), (R_c(x))^*) = \kappa(U, A, I)$.

Proof. By Theorem 5, $(R_c(x), (R_c(x))^*)$ is a concept of (U, A, I). The intent of each concept generated by $\bigvee(R_c(x), (R_c(x))^*)$ is contained in $\{x\}^*$, that is, the intents of all concepts, whether in the the original context or sub-context, are included in $R_c(x)'s$ attribute sets. The extent of concept on the sub-context is the extension on the original context, so the intent and extent obtained by $\bigvee$ belong to the original context. All concepts on the original context can be formed by $(R_c(x), (R_c(x))^*)$. □

5 CONCEPT GRANULAR DESCRIPTION BASED ON THE QUOTIENT SPACE

The quotient space provides a quick way of simplifying the complex system. But how to measure its effects? In this section, we will discuss granular entropy.

The entropy of a system as defined by Shannon [29], called information entropy, gives a measure of uncertainty about its actual structure. Unlike Shannon's entropy, the proposed entropy can measure not only uncertainty in information systems, but also a granular classification. For information system (U, A, V, f), a quotient space can be constituted by equivalence relation R of $B \subseteq A$. R can be seen as a map defined in granules based on the division function. So we can define the distribution function of the map and its information entropy.

Definition 10. Let (U, A, V, f) be an information system, $B \subseteq A$. R_B deduces quotient space $([u_i]_{R_B}, R, [A])$ with respect to U. The partition granule of (U, A, V, f) is defined by $X_i = [u_i]_{R_B} (i = 1, 2, \ldots, m)$. $\|[u_i]_{R_B}\|$ is its granular value. Its granularity is denoted as $G(X_i) = \|X_i\|/|U|$.

In general, the value of $\|X_i\|$ is 1.

Theorem 7. For $(U, A, V, f), B \subseteq A, R_B \in P(A)$, quotient space $([u_i]_{R_B}, R, [A])$ has the following properties:

1. $\cup_{u \in U}[u]_{R_B} = U$;
2. $[u_i]_{R_B} \cap [u_j]_{R_B} = \emptyset$, when $[u_i]_{R_B} \neq [u_j]_{R_B}$.

Proof. It is easy to see that the results are true according to Definition 10. □

Definition 11 (Granular entropy)**.** Let (U, A, V, f) be an information system. A granular entropy of U/R_B is defined as

$$E(R_B) = \sum_{i=1}^{m} G(X_i)/|U|.$$

For an information system (U, A, V, f), $U/R_A = \{X_1, X_2, \ldots, X_m\}$, Shannon defined an information entropy [29]: $H(A) = -\sum_{i=1}^{m} p_i log_2 p_i, p_i = \frac{|X_i|}{|U|}$. Liang [32] put forward a new definition of information entropy: $E(A) = \sum_{i=1}^{m} \frac{|X_i||X_i^c|}{|U||U|} = \sum_{i=1}^{m} \frac{|X_i|}{|U|}(1 - \frac{|X_i|}{|U|})$, where X_i^c denotes the complement set of X_i, i.e., $X_i^c = U - X_i$. $\frac{|X_i|}{|U|}$ represents the probability of X_i within the universe U and $\frac{|X_i^c|}{|U|}$ is the probability of the complement set of X_i within the universe U. This entropy can measure not only uncertainty in information systems, but also fuzziness of a rough set and a rough classification.

$E(R_B)$ is mainly used to identify divided universe along with the attributes of monotonic increasing or decreasing in knowledge B. The attributes of knowledge B gradually increasing results in the division of B to universe finer, produced grain size smaller. $E(R_B)$ increased gradually, knowledge B identification gradually increases. The classification is more accurate. And vice versa. If the increase or decrease of attributes with B, the division B to the universe is at constant and $E(R_B)$ are the same.

Theorem 8. Let $S = (U, A, V, f)$ be an information system, $B, D \subseteq A$. The following results hold in two quotient space with respect to R_B and R_D.

1. $E(R_B) = \sum_{i=1}^{m} \|X_i\|/|U|^2$;
2. $E(R_{B\cup D}) = \sum_{i,j} \|X_i \cap X_j\|/|U|^2$.

Proof. It is proved by Definitions 10 and 11. □

Theorem 9. Let (U, A, V, f) be an information system, $B, D \subseteq A$. If $U/R_B = U/R_D$, then $E(R_B) = E(R_D)$.

Proof. Since $U/R_B = U/R_D$, R_B and R_D deduce same partition on universe U, that is, $[u_i]_{R_B} = [u_i]_{R_D}(i = 1, 2, \ldots, m)$. They have the same quotient distribution. By the formula of the granular entropy, we can get $E(R_B) = E(R_D)$. □

This theorem states that knowledge have the same classification ability if they have same algebraic representations. However, the converse is not necessarily true.

Theorem 10. Let (U, A, V, f) be an information system, $B, D \subseteq A$. If $R_B \subseteq R_D \in P(A)$ and $E(R_B) = E(R_D)$, then $U/R_B = U/R_D$.

Proof. We assume that $U/R_B = \{X_1, X_2, \ldots, X_m\}, U/(R_D \setminus R_B) = \{Y_1, Y_2, \ldots, Y_m\}$. According to Definition 11 and Theorem 8 we can see that $E(R_B) = \sum_{i=1}^{m} \|X_i\|/|U|^2, E(R_B \cup (R_D \setminus R_B)) = \sum_{i,j} \|X_i \cap X_j\|/|U|^2$. $\|X_i\| = \sum_{j=1}^{m} \|X_i \cap X_j\|$ is available by $E(R_B) = E(R_D)$. Thus, there are two situations: $X_i \subset Y_j$ or that exists j_0, so that $\|X_i \cap X_{j_0}\|$, i.e. $X_i \cap X_{j_0} = \emptyset$. However, $\cup_j Y_j = U$ and $\|X_i \cap X_{j_0}\| = 0$, this will not happen. Hence, $X_i \subset Y_j$ for any X_i. Thus, $U/R_B = U/R_D$. □

The result given in Theorem 10 provides a theoretical foundation for knowledge reduction based on the entropy.

Theorem 11. Let (U, A, V, f) be an information system, $B \subseteq A$. Then $b \in B$ is unnecessary if and only if $E(R_{B-\{b\}}) = E(R_B)$.

Proof. The necessity is proved by Theorem 9. The sufficiency is proved by Theorem 10. □

Theorem 11 shows that adding an unnecessary attribute to a subset will not change the size of granule.

Corollary 1. Let (U, A, V, f) be an information system, $B \subseteq A$. Then $b \in B$ is unnecessary if and only if $E(R_{B-\{b\}}) > E(R_B)$.

Corollary 2. Let (U, A, V, f) be an information system, $B \subseteq A$. Then knowledge R_B is independent if and only if $E(R_{B-\{b\}}) > E(R_B), b \in B$.

Theorem 12 (Reduct)**.** Let (U, A, V, f) be an information system. $B \subseteq A$ is a reduct of knowledge R_B if and only if

1. $E(R_A) = E(R_B)$.
2. $E(R_{B-\{b\}}) < E(R_B)$ for $\forall b \in B$.

Proof. Suppose that B is a reduct of A, i.e., there is $U/R_A = U/R_B$ such that $E(R_A) = E(R_B)$ by Theorem 9. If $B - b \subseteq B$, exist $E(R_{B-\{b\}}) \geq E(R_B)$, then there is at least $U/R_{B-\{b\}} = U/R_B$ by Theorem 10. This contradicts with that B is a reduct. To show the converse, suppose that there is $E(R_A) = E(R_B)$, we can get $U/R_A = U/R_A$ by Theorem 10. Since $E(R_{B-\{b\}}) < E(R_B)$ for $\forall b \in B$, B is a reduct by Corollary 2. □

Example 2. An information system $S = (U, A, V, f)$, as shown in Table 2, where $U = \{1, 2, ...5\}, A = \{a, b, c, d, e\}$. Seek out a minimum reduct of A.

U	a	b	c	d	e
1	1	0	2	1	0
2	0	0	1	2	1
3	2	0	2	2	2
4	0	0	2	2	2
5	1	1	2	1	0

Table 2. An information system

By Theorem 8, $\{a\}^* = (\{1, 5\}, \{2, 4\}, \{3\})$, $E_{R_{\{a\}^*}} = 3/25$, $\{b\}^* = (\{1, 2, 3, 4\}, \{5\})$, $E_{R_{\{b\}^*}} = 2/25$, $\{c\}^* = (\{1, 3, 4, 5\}, \{2\})$, $E_{R_{\{c\}^*}} = 2/25$, $\{d\}^* = (\{1, 5\}, \{2, 3, 4\})$, $E_{R_{\{d\}^*}} = 2/25$, $\{e\}^* = (\{1, 5\}, \{2\}, \{3, 4\})$, $E_{R_{\{e\}^*}} = 3/25$, are calculated, and then get $E_{R_A} = 5/25$.

Similarly, we can get $E_{R_{\{a,b\}}} = 4/25$, $E_{R_{\{a,b,c\}}} = 5/25$, $E_{R_{\{a,b,c,d\}}} = 5/25$, $E_{R_{\{a,c\}}} = 4/25$, $E_{R_{\{a,d\}}} = 3/25$, $E_{R_{\{a,e\}}} = 4/25$, $E_{R_{\{b,c\}}} = 3/25$, $E_{R_{\{b,d\}}} = 3/25$, $E_{R_{\{b,e\}}} = 4/25$, $E_{R_{\{c,d\}}} = 3/25$, $E_{R_{\{c,e\}}} = 3/25$, $E_{R_{\{d,e\}}} = 3/25$, $E_{R_{\{a,b,d\}}} = 4/25$, $E_{R_{\{a,b,e\}}} = 5/25$, $E_{R_{\{a,c,d\}}} = 4/25$, $E_{R_{\{a,c,e\}}} = 4/25$, $E_{R_{\{a,d,e\}}} = 4/25$, $E_{R_{\{b,c,d\}}} = 4/25$, $E_{R_{\{b,c,e\}}} = 4/25$, $E_{R_{\{b,d,e\}}} = 4/25$, $E_{R_{\{c,d,e\}}} = 3/25$, $E_{R_{\{a,b,c,e\}}} = 5/25$, $E_{R_{\{a,b,c,d\}}} = 5/25$, $E_{R_{\{a,b,d,e\}}} = 5/25$, $E_{R_{\{b,c,d,e\}}} = 4/25$.

It is easy to see that $E_{R_{\{a,b,e\}}} = E_{R_{\{a,b,c\}}} = 5/25 = E_{R_A}$, so, $\{a, b, c\}$ and $\{a, b, e\}$ are the minimum reduct.

6 CONSTRUCTING A CONCEPT LATTICE BASED ON QUOTIENT SPACE

Quotient space allows us to construct concept lattices on different granules. In practical applications, it is not necessary to construct the concept lattice of all granules and merge them to get the original concept lattice. By selecting a few suitable granules, the original concept lattice can be obtained by merging.

Example 3. The formal context in Table 3 is a minor revision of the famous example, a film "Living Beings and Water" [2]. Since we require all the formal contexts in this paper are canonical, we delete the attribute a (water) from the original formal context. The objects are living beings mentioned in the film and are denoted by $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, where 1 is leech, 2 is bream, 3 is frog, 4 is dog, 5 is spike-weed, 6 is reed, 7 is bean, and 8 is maize. And the attributes in $A = \{b, c, d, e, f, g, h, i\}$ are the properties which the film emphasizes: b: lives in water, c: lives on land, d: needs chlorophyll to produce food, e: two seed leaves, f: one seed leaf, g: can move around, h: has limbs, and i: suckles its offspring.

U	b	c	d	e	f	g	h	i
1	×					×		
2	×					×	×	
3	×	×				×	×	
4		×				×	×	×
5	×		×		×			
6	×	×	×		×			
7		×	×	×				
8		×	×		×			

Table 3. Living beings and water (U, A, I)

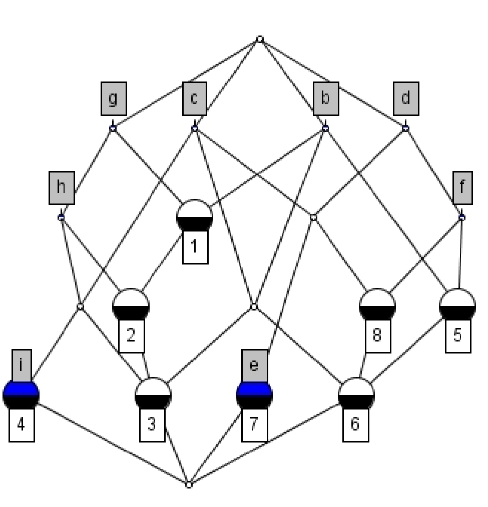

Figure 2. Concept lattice of Living Beings and Water $\kappa(U, A, I)$

The classified sets: $R_c(1) = \{1, 2, 3\}, R_c(2) = \{2, 3\}, R_c(3) = \{3\}, R_c(4) = \{4\}, R_c(5) = \{5, 6\}, R_c(6) = \{6\}, R_c(7) = \{7\}, R_c(8) = \{6, 8\}$.

The formal concepts determined by classified set $R_c(1) = \{1, 2, 3\}$ are: $(U, \emptyset)$, $(\{3\}, \{b, c, g, h\})$, $(\{2, 3\}, \{b, g, h\})$, $(\{1, 2, 3\}, \{b, g\})$, $(\{3, 4\}, \{c, g, h\})$, $(\{2, 3, 4\}, \{g, h\})$, $(\{1, 2, 3, 4\}, \{g\})$, $(\{3, 6\}, \{b, c\})$, $(\{3, 4, 6, 7, 8\}, \{c\})$, $(\{1, 2, 3, 5, 6\}, \{b\})$.

$R_c(4) = \{4\}$: $(U,\emptyset)$, $(\{4\},\{c,g,h,i\})$, $(\{3,4\},\{c,g,h\})$, $(\{2,3,4\},\{g,h\})$, $(\{1,2,3,4\},\{g\})$, $(\{3,4,6,7,8\},\{c\})$.

$R_c(5) = \{5,6\}$: $(U,\emptyset)$, $(\{6\},\{b,c,d,f\})$, $(\{5,6\},\{b,d,f\})$, $(\{6,8\},\{c,d,f\})$, $(\{6,7,8\},\{c,d\})$, $(\{5,6,8\},\{d,f\})$, $(\{5,6,7,8\},\{d\})$, $(\{1,2,3,5,6\},\{b\})$, $(\{3,4,6,7,8\},\{c\})$.

$R_c(7) = \{7\}$: $(U,\emptyset)$, $(\{7\},\{c,d,e\})$, $(\{6,7,8\},\{c,d\})$, $(\{5,6,7,8\},\{d\})$, $(\{3,4,6,7,8\},\{c\})$.

$R_c(8) = \{6,8\}$: $(U,\emptyset)$, $(\{6\},\{b,c,d,f\})$, $(\{5,6\},\{b,d,f\})$, $(\{6,8\},\{c,d,f\})$, $(\{6,7,8\},\{c,d\})$, $(\{5,6,8\},\{d,f\})$, $(\{5,6,7,8\},\{d\})$, $(\{1,2,3,5,6\},\{b\})$, $(\{3,4,6,7,8\},\{c\})$.

Since $R_c(1)\bigcup R_c(4)\bigcup R_c(5)\bigcup R_c(7)\bigcup R_c(8) = U$, $\bigcup(R_c(x),\ L(R_c(x))_{x\in\{1,4,5,7,8\}} = \kappa(U,A,I)$.

6.1 Algorithm

Main idea: Concept lattice is to be constructed in the simplified formal context. This concept lattice is a one-way homomorphism to the original one. The simplified concept lattice can be extended to restore according to the need.

Input: (U,A,I) where $U = \{u_1,u_2,\ldots,u_{|U|}\}, A = \{a_1,a_2,\ldots,a_{|A|}\}$

Output: Simplified (U',A',I')//To restore concept lattice of the original formal context by it.

For $i = 1$ to $|U|$ begin
  For $j = 1$ to $|A|$ begin
    If$(u_i,a_j)\in I$
    then $(u_i)^*_{a_j}\leftarrow a_j$
    else $(u_i)^*_{a_j}\leftarrow\emptyset$
  End
End
For $i = 1$ to $|A|$ begin
  For $j = 1$ to $|U|$ begin
    If$(u_i,a_j)\in I$
    then $(a_i)^*_{u_j}\leftarrow u_j$
    else $(a_i)^*_{u_j}\leftarrow\emptyset$
  End
End
For $i = 1$ to $|U|$ begin
  For $i = 1$ to $|U|$ begin
    $D_i\leftarrow\emptyset$
      If $(u_i)^*_{a_k}\supseteq(u_j)^*_{a_k}$
      then $D_i\leftarrow(u_i)^*_{a_k}\cup(u_j)^*_{a_k}$
    End
End
For $i = 1$ to $|A|$ begin

```
For i = 1 to |A| begin
   E_i ← ∅
      If (a_i)*_{u_l} ⊇ (a_j)*_{u_l}
      then E_i ← (a_i)*_{u_l} ∪ (a_j)*_{u_l}
   End
End
Reduced formal context (D, E, I) = (U′, A′, I′) with quotient space division.
To restore concept lattice κ(U, A, I) of the original formal context.
```

Example 4. Data table is shown in Table 4. It is a classic example of formal concept analysis [2].

		1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
		fur(hair)	feathers	scales	can fly	lives in water	lay eggs	produces milk	has a backbone	warm-blooded	cold-blooded	domestic
1	Bat	×			×			×	×	×		
2	Bear	×						×	×	×		
3	Cat	×						×	×	×		×
4	Chicken		×				×		×	×		×
5	Dog	×						×	×	×		×
6	Dolphin	×				×		×	×	×		
7	Elephant	×						×	×	×		
8	Frog					×	×		×		×	
9	Hawk		×		×		×		×	×		
10	Housefly	×			×		×				×	
11	Owl		×		×		×		×	×		
12	Sea lion	×				×		×	×	×		
13	Snake			×			×		×		×	
14	Spider	×					×				×	
15	Turtle			×			×		×		×	

Table 4. An illustrative example of a formal context of animals and their attributes

In order to distinguish between objects and attributes, we attach a letter subscript for each number. The uppercase is object, the lowercase is attribute.

Step 1.

$$
\begin{aligned}
\{1_f\}^* &= \{1_B, 2_B, 3_C, 5_D, 6_D, 7_E, 10_H, 12_S, 14_S\},\\
\{2_f\}^* &= \{4_C, 9_H, 11_O\},\\
\{3_s\}^* &= \{13_S, 15_T\},\\
\{4_f\}^* &= \{1_B, 9_H, 10_H, 11_O\},\\
\{5_w\}^* &= \{6_D, 8_F, 12_S\},\\
\{6_e\}^* &= \{4_C, 8_F, 9_H, 10_H, 11_O, 13_S, 14_S, 15_T\},\\
\{7_m\}^* &= \{1_B, 2_B, 3_C, 5_D, 6_D, 7_E, 12_S\},\\
\{8_b\}^* &= \{1_B, 2_B, 3_C, 4_C, 5_D, 6_D, 7_E, 8_F, 9_H, 11_O, 12_S, 13_S, 15_T\},\\
\{9_w\}^* &= \{1_B, 2_B, 3_C, 4_C, 5_D, 6_D, 7_E, 9_H, 11_O, 12_S\},\\
\{10_c\}^* &= \{4_C, 8_F, 10_H, 13_S, 14_S, 15_T\},\\
\{11_d\}^* &= \{3_C, 4_C, 5_D\}.\\
\{6_e\}^* - \{10_c\}^* &= \{4_C, 9_H, 11_O\},\\
\{10_c\}^* - \{3_s\}^* &= \{8_F, 10_H, 14_S\}.
\end{aligned}
$$

That is,

$$
\begin{aligned}
&\{3_s\}^* \subseteq \{8_b\}^*,\\
&\{3_s\}^* \subseteq \{10_c\}^* \subseteq \{6_e\}^*,\\
&\{11_d\}^* \subseteq \{9_w\}^* \subseteq \{8_b\}^*,\\
&\{7_m\}^* \subseteq \{9_w\}^* \subseteq \{8_b\}^*,\\
&\{2_f\}^* \subseteq \{9_w\}^*,\\
&\{2_f\}^* \subseteq \{6_e\}^*,\\
&\{5_w\}^* \subseteq \{8_b\}^*.
\end{aligned}
$$

So, $\{1_f\}^*$, $\{4_f\}^*$, $\{6_e\}^*$, $\{8_b\}^*$ are more coarse granules.

$$
\begin{aligned}
\{1_B\}^* &= \{1_f, 4_c, 7_m, 8_b, 9_w\},\\
\{2_B\}^* &= \{1_f, 7_m, 8_b, 9_w\},\\
\{3_C\}^* &= \{1_f, 7_m, 8_b, 9_w, 11_d\},\\
\{4_C\}^* &= \{2_f, 6_e, 8_b, 9_w, 11_d\},\\
\{5_D\}^* &= \{1_f, 7_m, 8_b, 9_w, 11_d\},\\
\{6_D\}^* &= \{1_f, 5_w, 7_m, 8_b, 9_w\},\\
\{7_E\}^* &= \{1_f, 7_m, 8_b, 9_w\},\\
\{8_F\}^* &= \{5_w, 6_e, 8_b, 10_c\},\\
\{9_H\}^* &= \{2_f, 4_f, 6_e, 8_b, 9_w\},\\
\{10_H\}^* &= \{1_f, 4_f, 6_e, 10_c\},\\
\{11_O\}^* &= \{2_f, 4_f, 6_e, 8_b, 9_w\},
\end{aligned}
$$

$$\begin{aligned}
\{12_S\}^* &= \{1_f, 5_w, 7_m, 8_b, 9_w\},\\
\{13_S\}^* &= \{3_s, 6_e, 8_b, 10_c\},\\
\{14_S\}^* &= \{1_f, 6_e, 10_c\},\\
\{15_S\}^* &= \{3_s, 6_e, 8_b, 10_c\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\{2_B\}^* &= \{7_E\}^* \subseteq \{3_C\}^* \subseteq \{1_B\}^*,\\
\{7_E\}^* &\subseteq \{12_S\}^* = \{6_D\}^*,\\
\{3_C\}^* &= \{5_D\}^*,\\
\{6_D\}^* &= \{12_S\}^*,\\
\{14_S\}^* &\subseteq \{10_H\}^*,\\
\{13_S\}^* &= \{15_T\}^*,\\
\{9_H\}^* &= \{11_O\}^*.
\end{aligned}$$

$\{1_B\}^*$, $\{4_C\}^*$, $\{5_D\}^*$, $\{6_D\}^*$, $\{8_F\}^*$, $\{9_H\}^*$, $\{10_H\}^*$, $\{13_S\}^*$ are coarse granules.

Step 2. A simplified formal context (U', A', I') constructed by quotient space is shown in Table 5.

		1	*4*	*6*	*8*
		fur(hair)	can fly	lay eggs	has a backbone
1	Bat	×	×		×
4	Chicken			×	×
5	Dog	×			×
6	Dolphin	×			×
8	Frog			×	×
9	Hawk		×	×	×
10	Housefly	×	×	×	
13	Snake			×	×

Table 5. A simplified formal context (U', A', I')

Concept lattice $\kappa'(U, A, I)$ of the simplified formal context is built as shown in Figure 3. It is easy to verify, $E_{R_{\{1_f, 4_f, 6_e, 8_b\}}} = 4/225$, $\{1_f, 4_f, 6_e, 8_b\}$ is the minimum reduct.

Step 3. According to (1), $\{3_s\}^* \subseteq \{10_c\}^* \subseteq \{6_e\}^*$, the concept that its attribute contains 6_e can be added 10_c but not objects 4_C, 9_H, 10_H. $(\{10_H\}, \{1_f, 4_f, 6_e, 10_c\})$ is deduced from $(\{10_H\}, \{1_f, 4_f, 6_e\})$. $\{11_d\}^* \subseteq \{9_w\}^* \subseteq \{8_b\}^*$: $(\{1_B, 9_H\},$

$\{4_f 8_b\}) \Rightarrow (\{1_B, 9_H\}, \{4_f, 8_b, 9_w\})$. $\{7_m\}^* \subseteq \{9_w\}^* \subseteq \{8_b\}^*$: $(\{1_B\}, \{1_f, 4_f, 8_b\}) \Rightarrow (\{1_B\}, \{1_f, 4_f, 8_b, 7_m, 9_w\})$.

According to (2), $\{2_B\}^* = \{7_E\}^* \subseteq \{1_B\}^*$, $\{6_D\}^* = \{12_S\}^*$, $\{9_H\}^* = \{11_O\}^*$, $\{13_S\}^* = \{15_T\}^*$: $(\{1_B, 4_C, 5_D, 6_D, 8_F, 9_H, 13_S\}, \{8_b\}) \Rightarrow (\{1_B, 2_B, 4_C, 5_D, 6_D, 7_E, 8_F, 9_H, 11_O, 12_S, 13_S, 15_T\}, \{8_b\})$. $\{9_H\}^* = \{11_O\}^*$: $(\{1_B, 9_H, 10_H\}, \{4_f\}) \Rightarrow (\{1_B, 9_H, 10_H, 11_O\}, \{4_f\})$. $\{9_H\}^* = \{11_O\}^*$, $\{13_S\}^* = \{15_T\}^*$: $(\{4_C, 8_F, 9_H, 13_S\}, \{6_e, 8_b\}) \Rightarrow (\{4_C, 8_F, 9_H, 11_O, 13_S, 15_T\}, \{6_e, 8_b\})$. $\{9_H\}^* = \{11_O\}^*$, $\{13_S\}^* = \{15_T\}^*$, $\{14_S\}^* \subseteq \{10_H\}^*$: $(\{4_C, 8_F, 9_H, 10_H, 13_S\}, \{6_e\}) \Rightarrow (\{4_C, 8_F, 9_H, 10_H, 11_O, 13_S, 14_S, 15_T\}, \{6_e\})$. $\{3_C\}^* = \{5_D\}^*$, $\{6_D\}^* = \{12_S\}^*$, $\{14_S\}^* \subseteq \{10_H\}^*$: $(\{1_B, 5_D, 6_D, 10_H\}, \{1_f\}) \Rightarrow (\{1_B, 3_C, 5_D, 6_D, 10_H, 12_S, 14_S\}, \{1_f\})$. $\{9_H\}^* = \{11_O\}^*$: $(\{9_H, 10_H\}, \{4_f, 6_e\}) \Rightarrow (\{1_B, 9_H, 11_O\}, \{4_f, 6_e\})$. $\{9_H\}^* = \{11_O\}^*$, $\{9_w\}^* \subseteq \{8_b\}^*$: $(\{1_B, 9_H\}, \{4_f, 8_b\}) \Rightarrow (\{1_B, 9_H, 11_O\}, \{4_f, 8_b, 9_w\})$. $\{2_B\}^* = \{7_E\}^* \subseteq \{1_B\}$, $\{14_S\}^* \subseteq \{10_H\}^*$: $(\{1_B, 10_H\}, \{1_f, 4_f,\}) \Rightarrow (\{1_B, 10_H\}, \{1_f, 4_f,\})$. $\{2_B\}^* = \{7_E\}^* \subseteq \{1_B\}$, $\{3_C\}^* = \{5_D\}^*$, $\{6_D\}^* = \{12_S\}^*$, $\{7_m\}^* \subseteq \{9_w\}^* \subseteq \{8_b\}^*$: $(\{1_B, 5_D, 6_D\}, \{1_f, 8_b,\}) \Rightarrow (\{1_B, 2_B, 3_C, 5_D, 6_D, 7_E, 12_S\}, \{1_f, 7_m, 8_b, 9_w\})$. $\{9_H\}^* = \{11_O\}^*$, $\{2_f\}^* \subseteq \{6_e\}^*$, $\{9_w\}^* \subseteq \{8_b\}^*$: $(\{9_H\}, \{4_f, 6_e, 8_b\}) \Rightarrow (\{9_H, 11_O\}, \{2_f, 4_f, 6_e, 9_w\})$.

In addition, according to (1), we can obtain the following results. $\{3_s\}^* \cap \{6_e\}^* = \{3_s\}^* \cap \{8_b\}^* = \{3_s\}^* \cap \{10_c\}^* = \{13_S, 15_T\} \Rightarrow (\{13_S, 15_T\}, \{3_s, 6_e, 8_b, 10c\})$. $\{6_e\}^* \cap \{10_c\}^* = \{8_F, 10_H, 13_S, 14_S, 15_T\} \Rightarrow (\{8_F, 10_H, 13_S, 14_S, 15_T\}, \{6_e, 10c\})$. $\{8_b\}^* \cap \{9_w\}^* = \{1_B, 2_B, 3_C, 4_C, 5_D, 6_D, 7_E, 9_H, 11_O, 12_S\} \Rightarrow (\{1_B, 2_B, 3_C, 4_C, 5_D, 6_D, 7_E, 9_H, 11_O, 12_S\}, \{8_b, 9w\})$. $\{8_b\}^* \cap \{11_d\}^* = \{9_w\}^* \cap \{11_d\}^* = \{3_C, 4_C, 5_D\} \Rightarrow (\{3_C, 4_C, 5_D\}, \{8_b, 9w, 11_d\})$. $\{5_w\}^* \cap \{8_b\}^* = \{6_D, 8_F, 12_S\} \Rightarrow (\{6_D, 8_F, 12_S\}, \{5_w, 8_b\})$. $\{2_f\}^* \cap \{6_e\}^* = \{2_f\}^* \cap \{8_b\}^* = \{4_C, 9_H, 11_O\} \Rightarrow (\{4_C, 9_H, 11_O\}, \{2_f, 6e, 8_b\})$.

By (2), we get that $\{3_C\}^* \cap \{5_D\}^* = \{1_f, 7_m, 8_b, 9_w, 11_d\} \Rightarrow (\{3_C, 5_D\}, \{1_f, 7_m, 8_b, 9_w, 11_d\})$. $\{6_D\}^* \cap \{12_S\}^* = \{1_f, 5_w, 7_m, 8_b, 9_w\} \Rightarrow (\{6_D, 12_S\}, \{1_f, 5_w, 7_m, 8_b, 9_w\})$. $\{12_S\}^* \cap \{14_S\}^* = \{1_f, 6_e, 10_c\} \Rightarrow (\{12_S, 14_S\}, \{1_f, 6_e, 10_c\})$.

In the new context, object 5, 6, and 4, 8, 13 have the same attributes respectively, so we should consider the possibility that these objects form concepts. $\{5_D\}^* \cap \{6_D\}^* = \{1_f, 7_m, 8_b, 9_w\} \Rightarrow (\{1_B, 2_B, 3_C, 5_D, 6_D, 7_E, 12_S\}, \{1_f, 7_m, 8_b, 9_w\})$. The concept already exists. At the same time, $\{3_C\}^* = \{5_D\}^*$, $\{6_D\}^* = \{12_S\}^*$, object 5 and 6 cannot construct the concept. $\{4_C\}^* \cap \{8_F\}^* = \{4_C\}^* \cap \{13_S\}^* = \{6_e, 8_b\} \Rightarrow (\{4_C, 8_F, 9_H, 11_O, 13_S, 15_T\}, \{6_e, 8_b\})$. The concept already exists. $\{13_S\}^* = \{15_T\}^*$, object 13 cannot construct the concept. $\{13_S\}^* = \{15_T\}^*$, $\{8_F\}^* \cap \{13_S\}^* = \{6_e, 8_b, 10_c\} \Rightarrow (\{8_F, 13_S, 15_T\}, \{6_e, 8_b, 10_c\})$. Finally, calculate the two single object concepts, $(\{4_C\}, \{2_f, 6_e, 8_b, 9_w, 11_d\})$, $(\{8_F, \{5_w, 6_e, 8_b, 10_c\})$.

Step 4. To construct the concept lattice $\kappa(U, A, I)$ as shown in Figure 4.

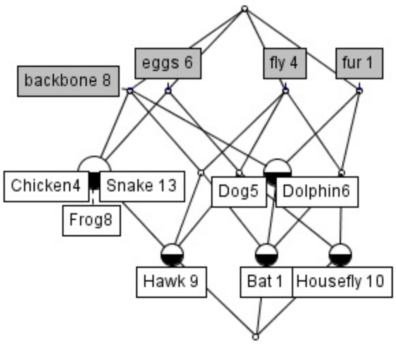

Figure 3. Concept lattice $\kappa'(U, A, I)$

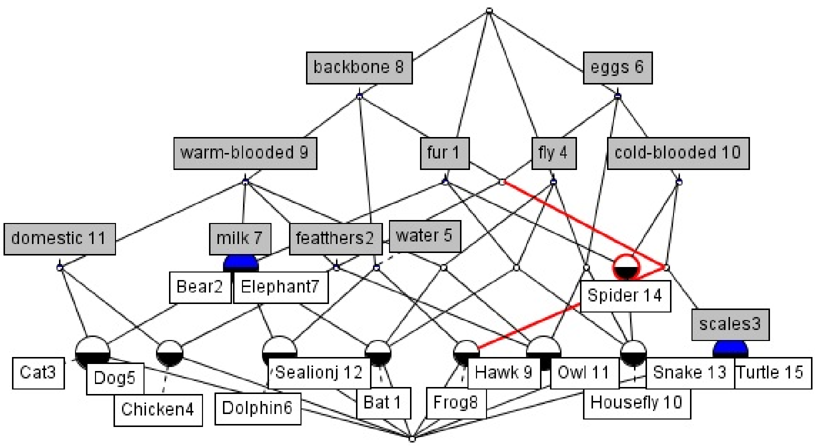

Figure 4. Concept lattice $\kappa(U, A, I)$

6.2 Experiments

Denote $N = min\{|U|, |A|\}$, we know the time complexity of Step 1 in Algorithms is $O(N^2)$. So we can get two matters as follows.

1. The time complexity of algorithm is $O(4N^2)$.
2. Suppose that Algorithm will be terminated in the k^{th} step; then the time complexity of Algorithm is $O\left(\sum_{i=1}^{i=k}(N \times k)\right)$. We can easily get $O\left(\sum_{i=1}^{i=k}(N \times k)\right) \leq O\left(4N^2\right)$.

To verify the effectiveness of the algorithm, we compared it with Godin algorithm [24] (Basic Incremental Update Algorithms) using a server that contains a Intel® Core™ i5-4590 3.30 GHz CPU, 4 GB memory, Windows 7 operating system and Visual C++ 6.0. The experimental data is random. The number of attributes is set to 20. The number of objects is increased from 50 to 1 050. Change interval is 50. Figure 5 shows calculation results of two kinds of algorithms.

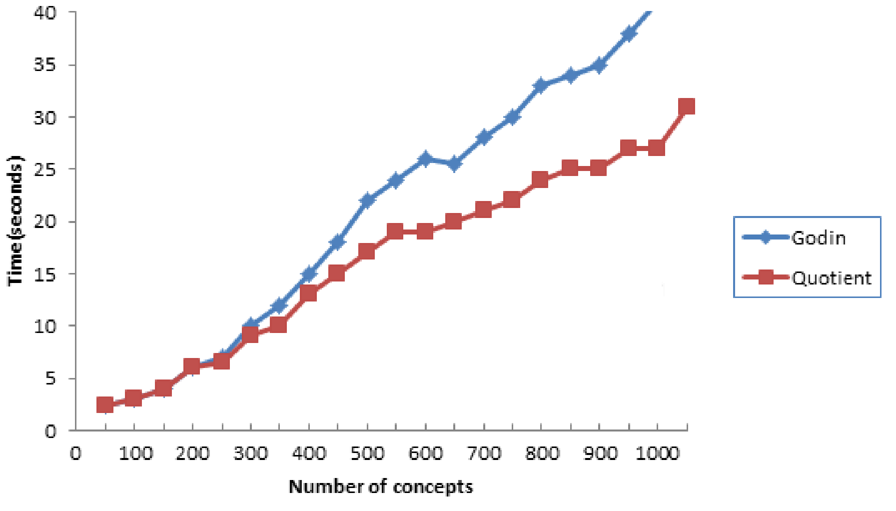

Figure 5. Comparison of the proposed algorithm with the Godin algorithm

In Figure 5, the abscissa denotes the number of formal concept. The ordinate is time to calculate the concept lattice. As it can be seen, the number is less than 200, the difference is not obvious. The proposed algorithm is better when the number becomes larger. It is proved that the proposed algorithm is effective.

It should be noted that the proposed algorithm is based on the quotient space granules. The algorithm has no obvious advantage in speed, if the information system can only be divided into a finer granule. It is, in fact, that the existing data space is abstracted into a "coarse grains" one. Thereby, the dimension of data is reduced, also simplifies the concept lattice. This simplified process can be thought of as hidden knowledge at different level, and it is suited to the needs of the problem analysis. The comparison of speed, in a sense, is not the most important.

7 CONCLUSION AND FUTURE WORK

This paper introduces concept information granule, granular entropy and quotient space into concept lattice research, and presents a unified research model for expansion and reduction of concept lattice in different granulations, and provides a detailed description of this overall process. In this model, it mainly obtains conclusions as follows:

1. A concept information granule is provided for the concept lattice research. As the knowledge in basic level, the concept information granule not only offers a uniform technology for concept learning on the whole, but it is also convenient for knowledge sharing and reuse in different levels;
2. The granular quotient space is introduced to lattice building research, which can overcome the impact on the application of FCA caused by the time complexity and space complexity problem to some extent, it helps to find useful information and avoids users getting lost in the complex information;
3. A new granular entropy between concepts is given in different granulations, which can help experts judge relations except for inheritance relation and measure the degree of reduction of the context.

Although the FCA based on granular quotient space proposed in this paper is only a starting point and a lot of subsequent study is needed, it offers a new way or guideline for the concept lattice reduction. How to combine concept lattice with quotient space more rationally and reduce human judgments is one focus of our research in the future.

REFERENCES

[1] Dai, J.—Song, J.—Wang, G.: A Rapid Concept Granules Extraction Algorithm Based on Cloud Model. Journal of Computational Information Systems, Vol. 8, 2012, No. 20, pp. 8413–8420.

[2] Ganter, B.—Wille, R.: Formal Concept Analysis: Mathematical Foundations. Springer-Verlag New York, Inc., 1999.

[3] Pal, S. K.—Meher, S. K.: Natural Computing: A Problem Solving Paradigm with Granular Information Processing. Applied Soft Computing, Vol. 13, 2013, No. 9, pp. 3944–3955.

[4] Poelmans, J.—Elzinga, P.—Viaene, S.—Dedene, G.: Formal Concept Analysis in Knowledge Discovery: A Survey. In: Croitoru, M., Ferré, S., Lukose, D. (Eds.): Conceptual Structures: From Information to Intelligence (ICCS 2010). Springer, Berlin, Heidelberg, Lecture Notes in Computer Science, Vol. 6208, 2010, pp. 139–153.

[5] Qian, Y. H.—Zhang, H.—Li, F. J.—Hu, Q. H.—Liang, J. Y.: Set-Based Granular Computing: A Lattice Model. International Journal of Approximate Reasoning, Vol. 55, 2014, No. 3, pp. 834–852.

[6] Qiu, G. F.—Chen, J.: Concept Knowledge System and Concept Information Granular Lattice. Chinese Journal of Engineering Mathematics, Vol. 22, 2005, No. 6, pp. 963–969.

[7] Qiu, G. F.—Ma, J. M.—Yang, H. Z.—Zhang, W. X.: A Mathematical Model for Concept Granular Computing Systems. Science China Information Sciences, Vol. 53, 2010, No. 7, pp. 1397–1408.

[8] Ding, S. F.—Li, J. Y.—Xu, L.—Qian, J.—Zhao, X. W.—Jin, F. F.: Research Progress of Granular Computing (GrC). International Journal of Digital Content Technology and Its Applications, Vol. 5, 2011, No. 1, pp. 162–172.

[9] Wang, L. D.—Liu, X. D.—Wang, X.: AFS-Based Formal Concept Analysis within the Logic Description of Granules. In: Yao, J. et al. (Eds.): Rough Sets and Current Trends in Computing (RSCTC 2012). Springer, Berlin, Heidelberg, Lecture Notes in Computer Science, Vol. 7413, 2012, pp. 323–331.

[10] Wille, R.: Restructuring Lattice Theory: An Approach Based on Hierarchies of Concepts. In: Rival, I. (Ed.): Ordered Sets. Springer, Dordrecht, NATO Advanced Study Institutes Series (Series C – Mathematical and Physical Sciences), Vol. 83, 2009, pp. 445–470.

[11] Wu, W. Z.: Attribute Granules in Formal Contexts. In: An, A., Stefanowski, J., Ramanna, S., Butz, C. J., Pedrycz, W., Wang, G. (Eds.): Rough Sets, Fuzzy Sets, Data Mining and Granular Computing (RSFDGrC 2007). Springer, Berlin, Heidelberg, Lecture Notes in Computer Science, Vol. 4482, 2007, pp. 395–402.

[12] Wu, W. Z.—Leung, Y.—Mi, J. S.: Granular Computing and Knowledge Reduction in Formal Contexts. IEEE Transactions on Knowledge and Data Engineering, Vol. 21, 2009, No. 10, pp. 1461–1474.

[13] Weihua, X. U.—Liu, S. H.: Granularity Representation of Knowledge in Information System Based on General Binary-Relation. Computer Engineering and Applications, Vol. 109, 2011, No. 2, pp. 619–630.

[14] Yao, Y. Y.: Concept Lattices in Rough Set Theory. IEEE Annual Meeting of the Fuzzy Information, 2004. Processing NAFIPS '04, Vol. 2, 2004, pp. 796–801.

[15] Yao, Y. Y.—Zhang, N.—Miao, D. Q.—Xu, F. F.: Set-Theoretic Approaches to Granular Computing. Fundamenta Informaticae, Vol. 115, 2012, No. 2-3, pp. 247–264.

[16] Zhang, B.—Zhang, L.: Theory and Applications of Problem Solving. Elsevier Science Inc. Press, pp. 770, 1992.

[17] Zhang, L.—Zhang, B.: Quotient Space Based Problem Solving: A Theoretical Foundation of Granular Computing. Qinghua University Press, 2014.

[18] Zhang, L.—Zhang, B.: The Quotient Space Theory of Problem Solving. In: Wang, G., Liu, Q., Yao, Y., Skowron, A. (Eds.): Rough Sets, Fuzzy Sets, Data Mining, and Granular Computing (RSFDGrC 2003). Springer, Berlin, Heidelberg, Lecture Notes in Computer Science, Vol. 2639, 2003, pp. 11–15.

[19] Zhang, Q.—Xing, Y.: Formal Concept Analysis Based on Granular Computing. Journal of Computational Information Systems, Vol. 6, 2010, No. 7, pp. 2287–2296.

[20] Zhang, W. X.—Yang, H. Z.—Ma, J. M.—Qiu, G. F.: Concept Granular Computing Based on Lattice Theoretic Setting. In: Bargiela, A., Pedrycz, W. (Eds.): Human-Centric Information Processing Through Granular Modelling. Springer, Berlin, Heidelberg, Studies in Computational Intelligence, Vol. 182, 2009, pp. 67–94.

[21] Wang, H.—Zhang, W. X.: Relationships Between Concept Lattice and Rough Set. In: Rutkowski, L., Tadeusiewicz, R., Zadeh, L. A., Żurada, J. M. (Eds.): Artificial Intelligence and Soft Computing – ICAISC 2006. Springer, Berlin, Heidelberg, Lecture Notes in Computer Science, Vol. 4029, 2006, pp. 538–547.

[22] Zhang, Q. H.—Wang, G. Y.—Liu, X. Q.: Rule Acquisition Algorithm Based on Maximal Granule. Pattern Recognition and Artificial Intelligence, Vol. 25, 2012, No. 3, pp. 388–396 (in Chinese).

[23] Lin, Y.—Zhang, C. P.: Semantic Analyses of Rough Truth for Axioms in Modal Logic. Journal of Computer Research and Development, Vol. 43, 2006, No. 11, pp. 1999–2004.

[24] Godin, R.—Missaoui, R.—Alaoui, H.: Incremental Concept Formation Algorithms Based on Galois (Concept) Lattices. Computational Intelligence, Vol. 11, 1995, No. 2, pp. 246–267.

[25] Schultz, B. R.: Lecture Notes on Point-Set Topology. Unpublished, 2012.

[26] Pawlak, Z.: Rough Sets: Theoretical Aspects of Reasoning about Data. Kluwer Academic Publishers Press, 1992.

[27] Pawlak, Z.: Information Systems Theoretical Foundations. Information Systems, Vol. 6, 1981, No. 3, pp. 205–218.

[28] Pawlak, Z.—Skowron, A.: Rudiments of Rough Sets. Information Sciences, Vol. 177, 2007, No. 1, pp. 3–27.

[29] Shannon, C. E.—Weaver, W.—Wiener, N.: The Mathematical Theory of Communication. University of Illinois Press, 1998.

[30] Zhang, L.—Zhang, B.: Problem Solving Theory and Application: Granularity Computing of Quotient Space Theory and Its Application. Tsinghua University Press, 2007.

[31] Xiong, J. C.: Point Set Topology Notes. Higher Education Press, 2011.

[32] Liang, J. Y.—Qian, Y. H.: Information Granules and Entropy Theory in Information Systems. Science China Information Sciences, Vol. 51, 2008, No. 10, pp. 1427–1444.

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