Computing and Informatics, Vol. 25, 2006, 393–404

ANALYSIS OF GREEDY ALGORITHM FOR VERTEX COVERING OF RANDOM GRAPH BY CUBES

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Manuscript received 4 November 2005; revised 12 June 2006
Communicated by Imre J. Rudas

Abstract. We study randomly induced subgraphs G of a hypercube. Specifically, we investigate vertex covering of G by cubes. We instantiate a greedy algorithm for this problem from general hypergraph covering algorithm [9], and estimate the length of vertex covering of G. In order to obtain this result, a number of theoretical parameters of randomly induced subgraph G were estimated.

Keywords: Random graphs, vertex covering, greedy algorithm

1 INTRODUCTION

Randomly induced subgraphs of the cube are related to minimization of Boolean functions in the class of disjunctive normal forms. A survey of results is presented in work by Kostochka, Sapozhenko, and Weber [7].

We study randomly induced subgraph of an n-cube. The model of random subgraphs is the following one: each edge is present in a subgraph with probability p ($0 < p < 1$), independently of the presence of other edges. Burtin [3] studied connectedness of these graphs for $p \neq \frac{1}{2}$. Erdös and Spencer [4] and Toman [10] studied components of random subgraphs for $p = \frac{1}{2}$. Ajtai, Komlós, and Szemeredi [1] analyzed components for $p = \frac{1+\varepsilon}{n}$. Toman [11] estimated radius of a random subgraph of an n-cube for $p \geq \frac{1}{2}$. Bollobas [2] and Kostochka [8] studied perfect matchings for $p \geq \frac{1}{2}$.

We show that random graph does not contain cubes of arbitrary sizes. We estimate the number of cubes of given order (we obtain an asymptotically precise estimates for some cube orders). Further, we estimate the number of cubes containing fixed vertex, and the relative number of cubes with this property.

These results are used to estimate the size of vertex covering of random graph by cubes obtained by greedy algorithm. The greedy algorithm was instantiated from hypergraph covering algorithm proposed by Sapozhenko [9]. As a corollary, we obtain an upper bound of vertex covering of random graph by cubes.

2 PRELIMINARIES

Let G be a graph. The vertex set of G will be denoted by $V(G)$, and the edge set by $H(G)$.

Let Q_n be an n-cube graph consisting of 2^n vertices labelled by binary vectors of length n, and $n2^{n-1}$ edges joining vertices differing in exactly one coordinate. We denote by G^n the set of all subgraphs of Q_n with the complete set of vertices. Thus, every $G \in G^n$ has 2^n vertices.

A random graph is a graph obtained from Q_n by independent removal of edges. The probability that the edge is not removed is denoted by p, where p is a constant ($0 < p < 1$). We shall consider a probabilistic space (model) (G^n, P), where $P : G^n \to \langle 0, 1\rangle$ is a probabilistic function defined as follows:

$$P(G) = p^{|H(G)|}(1-p)^{|H(Q_n)|-|H(G)|}.$$

The probabilistic function P can be naturally extended to arbitrary subset R of G^n:

$$P(R) = \sum_{G \in R} P(G).$$

We call a subset $R \subseteq G^n$ a property of graphs. We shall say that random graph has a property R, if $\lim_{n\to\infty} P(R) = 1$.

A graph K is a subgraph of G, denoted by $K \subseteq G$, if $V(K) \subseteq V(G)$ and $H(K) \subseteq H(G)$. For graphs $G \in G^n$, we shall say that K is contained in G, if $K \subseteq G$.

A real-valued random variable X is a measurable real-valued function on a probability space, $X : (G^n, P) \to R$. All random variables in this paper are non-negative integer random variables. Let X be a random variable. The expectation, and the variance of the random variable X will be denoted by $\mathrm{E}(X)$ and $\mathrm{Var}(X)$, respectively. The variance of a random variable X can be expressed as follows: $\mathrm{Var}(X) = \mathrm{E}(X^2) - \mathrm{E}(X)^2$.

Let X be a non-negative random variable and let $t > 0$. Then we have (Markov's inequality):

$$\Pr[X \geq t \cdot \mathrm{E}(X)] \leq \frac{1}{t}.$$

Now, let X be a real-valued random variable and let $d > 0$.Then we have (Chebyshev's inequality):

$$\Pr[|X - \mathrm{E}(X)| \geq d] \leq \frac{\mathrm{Var}(X)}{d^2}.$$

We say that a_n is asymptotically equal to b_n, notation $a_n \sim b_n$, if $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. The symbol $\ln x$ and $\lg x$ denotes natural and binary logarithm of x, respectively. We shall often use the logarithm to the base $\frac{1}{p}$. To simplify the notation, we put $b = \frac{1}{p}$ and write $\log_b x$ instead of $\log_{1/p} x$.

3 GREEDY COVERING OF THE RANDOM GRAPH BY CUBES

Let $V = \{v_1, \ldots, v_n\}$ be a finite set. Let $H = \{H_1, \ldots, H_m\}$ be a set of subsets of the set V such that $\bigcup_{i=1}^{m} H_i = V$. The pair (V, H) is called hypergraph with the set of vertices V, and the set of edges H. Degree of vertex $v_i \in V$ (denoted by $\deg(v_i)$) is the number of edges containing v_i.

Let (V, H) be a hypergraph. Let $V' \subseteq V$ and $H' \subseteq H$ be such subsets that $V' \subseteq \bigcup_{H_i \in H'} H_i$. We say that the set H' covers the set V', and the pair (V', H') is subhypergraph of (V, H) (denoted by $(V', H') \subseteq (V, H)$). If $V' = V$, the set H' is the covering of hypergraph (V, H), and $|H'|$ is the length of the covering.

Let ComputeCovering be a straightforward greedy algorithm for computation of covering, see Figure 1. The input of ComputeCovering is the hypergraph (V, H). The algorithm chooses one edge from H in every iteration. In the first iteration, the edge of maximal size is chosen. Let $H(k)$ denote the set of edges chosen in first k iterations of ComputeCovering. Let $V(k) = V \setminus \bigcup_{H_i \in H(k)} H_i$. If $V(k) = \emptyset$, the algorithm stops with result $H(k)$. Otherwise, the algorithm selects the edge H_w from $H \setminus H(k)$ containing largest number of vertices from $V(k)$. The algorithm sets $H(k+1) = H(k) \cup \{H_w\}$.

```
ComputeCovering(V,H)
k = 0;
H(0) = ∅; V(0) = V;
while V(k) ≠ ∅
    find edge H_w ∈ H \ H(k), such that |H_w| is maximal;
    H(k+1) = H(K) ∪ {H_w};
    V(k+1) = V \ ⋃_{H_i∈H} H_i;
    k = k+1;
end
return H(k)
```

Fig. 1. Algorithm ComputeCovering

It can be seen easily that ComputeCovering terminates with covering of hypergraph (V, H). We call this covering simple covering and its length is denoted by

$l_{\mathcal{A}}(V, G)$. The following theorem is proved in [9].

Theorem 1. Let (V, H) be a hypergraph. Let $(V', H') \subseteq (V, H)$ be a subhypergraph satisfying the following conditions:

1. $\forall v \in V'$: $\deg(v) \geq d$, for some integer d,
2. $|V'| \geq (1 - \varepsilon)|V|$, for some $\varepsilon \geq 0$.

Then the length of simple covering satisfies the inequality

$$l_{\mathcal{A}}(V, H) \leq 1 + \varepsilon|V| + \frac{|H'|}{d} \ln \frac{|V|de}{|H'|}.$$

We apply the theorem for obtaining upper bound of the length of vertex covering of random graph with 2^n vertices by cubes. For every graph $G = (V, H)$, a subgraph of an n-cube, a hypergraph (V_G, H_G) is defined as follows: $V_G = V$, and H_G is the set of all cubes contained in G. Let $H_{n,k}(G)$ be the set of all k-cubes contained in G. Let $\deg_k(v, G)$ be the number of k-cubes from $H_{n,k}(G)$ that contain vertex v. Further, let $V_G(k, d) \subseteq V$ be the subset of those vertices v, such that $\deg_k(v, G) \geq d$. Obviously $(V_G(k, d), H_{n,k}(G)) \subseteq (V_G, H_G)$. If $|V_G(k, d)| \geq (1 - \varepsilon)2^n$, for some $\varepsilon \geq 0$, then Theorem 1 can be applied to estimate the length of simple covering. We get

$$l_{\mathcal{A}}(V_G, H_G) \leq 1 + \varepsilon 2^n + \frac{|H_{n,k}(G)|}{d} \ln \frac{2^n de}{|H_{n,k}(G)|}.$$

In the following sections we prove some properties of random graphs to fill gaps in this upper bound. All properties are studied asymptotically, i.e. with probability converging to 1 for $n \to \infty$.

4 CUBES

We start with exploration of subcubes in random graph $G \in G^n$.

Definition 1. A subcube of order k, or k-subcube (for $0 \leq k \leq n$), is a k-cube subgraph of Q_n.

Recall that all random variables in this paper are random variables in probabilistic space (G^n, P). Let $X_{n,k}$ be a random variable denoting the number of k-subcubes contained in $G \in G^n$. Thus, $X_{n,k} = |H_{n,k}(G)|$. In subsequent lemmas we express the expected value of $X_{n,k}$ and estimate its variance.

Lemma 1. Let K be a k-subcube. Let $G \in G^n$. Then the probability that K is contained in G is

$$\Pr[K \subseteq G] = p^{k2^{k-1}}.$$

Proof.

$$\Pr[K \subseteq G] = p^{|H(K)|} = p^{k2^{k-1}}.$$

□

Lemma 2.

$$\mathrm{E}(X_{n,k}) = \binom{n}{k} 2^{n-k} p^{k2^{k-1}}.$$

Proof. The expected value can be expressed as a sum of probabilities $\Pr[K \subseteq G]$ over all k-subcubes of Q_n:

$$\mathrm{E}(X_{n,k}) = \sum_K \Pr[K \subseteq G].$$

Using the previous lemma we obtain

$$\mathrm{E}(X_{n,k}) = \sum_K p^{k2^{k-1}} = \binom{n}{k} 2^{n-k} p^{k2^{k-1}}.$$

□

Lemma 3.

$$\mathrm{Var}(X_{n,k}) = \binom{n}{k}^2 2^{n-k} p^{k2^{k-1}}.$$

Proof. We express $\mathrm{E}(X_{n,k}^2)$, the expected number of ordered pairs (K, L), such that K, L are k-subcubes, and $K, L \subseteq G$:

$$\mathrm{E}(X_{n,k}^2) = \sum_{(K,L)} p^{|H(K)\cup H(L)|} = \sum_{(K,L)} p^{|H(K)|+|H(L)|-|H(K\cap L)|} = p^{k2^k} \sum_{(K,L)} p^{-|H(K\cap L)|} \tag{1}$$

where the sum is taken over all k-subcubes of Q_n. If $K \cap L \neq \emptyset$, then $K \cap L$ a subcube of order j, for some $0 \leq j \leq k$. Thus, $|H(K \cap L)| = j2^{j-1}$. Let A_j count those pairs (K, L), such that $K \cap L$ is a j-subcube. Trivially, $A_j = \binom{n}{j} 2^{n-j} \binom{n-j}{k-j}\binom{n-k}{k-j}$. The number of pairs (K, L) such that $K \cap L = \emptyset$ is $\left(\binom{n}{k}2^{n-k}\right)^2 - \sum_{j=0}^{k} A_j$. Substitution in (1) gives

$$\begin{aligned} \mathrm{E}(X_{n,k}^2) &= p^{k2^k} \left(\sum_{j=0}^{k} A_j p^{-j2^{j-1}} + \left(\left(\binom{n}{k} 2^{n-k} \right)^2 - \sum_{j=0}^{k} A_j \right) p^0 \right) \\ &= p^{k2^k} 2^n \sum_{j=0}^{k} \binom{n}{j} \binom{n-j}{k-j} \binom{n-k}{k-j} 2^{-j} \left(p^{-j2^{j-1}} - 1 \right). \end{aligned}$$

We denote $a_j = 2^{-j}\left(p^{-j2^{j-1}} - 1\right)$, for $0 \leq j \leq k$. For estimating an upper bound of $\mathrm{Var}(X_{n,k})$ we use the inequality $a_k \geq a_j$, and following combinatorial identity, see [6]:

$$\sum_{j=0}^{k} \binom{n}{j} \binom{n-j}{k-j} \binom{n-k}{k-j} = \binom{n}{k}^2.$$

We have

$$\begin{aligned}\mathrm{Var}(X_{n,k}) &\le p^{k2^k}\cdot 2^n a_k \sum_{j=0}^{k}\binom{n}{j}\binom{n-j}{k-j}\binom{n-k}{k-j}\\ &= p^{k2^k}\cdot 2^n a_k\binom{n}{k}^2 = p^{k2^k}\cdot 2^{n-k}\left(p^{-k2^{k-1}}-1\right)\binom{n}{k}^2\\ &\le \binom{n}{k}^2 2^{n-k}p^{k2^k}.\end{aligned}$$

□

Theorem 2 (counting cubes of given order)**.** Let $\varphi(n)$ be an arbitrary increasing function. Then, with probability converging to 1 as $n \to \infty$, the following inequality holds for any $G \in G^n$:

$$\binom{n}{k}\left(2^{n-k}p^{k2^{k-1}}-\varepsilon\right) < X_{n,k} < \binom{n}{k}\left(2^{n-k}p^{k2^{k-1}}+\varepsilon\right),$$

where $\varepsilon = \varphi(n)\sqrt{2^{n-k}p^{k2^{k-1}}}$.

Proof. We substitute the results of Lemma 2 and Lemma 3 into Chebyshev inequality for random variable $X_{n,k}$. Moreover, we set $\varepsilon = \varphi(n)\binom{n}{k}\sqrt{2^{n-k}p^{k2^{k-1}}}$. Since $\lim_{n\to\infty} 1/\varphi(n) = 0$ we get

$$\lim_{n\to\infty}\Pr[|X_{n,k}-\mathrm{E}(X_{n,k})|\ge\varepsilon] \le \lim_{n\to\infty}\frac{\mathrm{Var}(X_{n,k})}{\varepsilon^2} = 0.$$

Hence

$$\lim_{n\to\infty}\Pr[|X_{n,k}-\mathrm{E}(X_{n,k})|<\varepsilon] = 1.$$

□

The maximal order of cubes contained in a random graph is an important quantity needed for the estimate of simple covering length. Let μ be the smallest integer satisfying the inequality

$$\mu - 1 + \lg(\mu+1) \ge \lg\log_b 2^n. \tag{2}$$

We show in Lemma 4 that random graph does not contain k-cubes, for any $k > \mu$.

Since μ is the smallest integer satisfying (2), we get

$$\mu - 2 + \lg\mu < \lg\log_b 2^n. \tag{3}$$

Inequalities (2) and (3) yield the following upper and lower bounds:

$$\left(\frac{1}{p}\right)^{(\mu+1)2^{n-1}} \geq 2^n \tag{4}$$

$$\left(\frac{1}{p}\right)^{\mu 2^{\mu-2}} < 2^n \tag{5}$$

For sufficiently large n we use the following estimates of μ:

$$\mu \leq \lceil \lg \log_b 2^n - \lg \lg \log_b 2^n + 2 \rceil \tag{6}$$

$$\mu \sim \lg \log_b 2^n = \lg n - \lg \lg \frac{1}{p}. \tag{7}$$

Lemma 4. The following statements hold for any $G \in G^n$, with probability converging to 1 as $n \to \infty$:

1. G does not contain cube of order greater than μ;
2. $X_{n,k} \sim \binom{n}{k} 2^{n-k} p^{k2^{k-1}}$, for $k \leq \mu - 2$.

Proof. In order to prove the first part, we show that $\lim_{n\to\infty} X_{n,k} = 0$, for any $k > \mu$. According to Theorem 2, it suffices to show that $\lim_{n\to\infty} \binom{n}{k}\varphi(n)2^{n-k}p^{k2^{k-1}} = 0$. Let $m_k = \binom{n}{k}\varphi(n)2^{n-k}p^{k2^{k-1}}$. The inequality $\binom{n}{k} \geq n^k$ yields $m_k \leq \varphi(n)2^{k\lg n+n-k}p^{k2^{k-1}}$. We can write $\mu = k + r$, for $r \geq 1$. Then

$$m_k \leq \varphi(n)2^{(\mu+r)\lg n}2^n p^{(\mu+r)2^{\mu+r-1}}.$$

Since $r \geq 1$ we get

$$m_k \leq \varphi(n)2^{(\mu+r)\lg n}2^n \left(p^{(\mu+r)2^{\mu-1}}\right)^{2^r}.$$

Using inequality (4) gives

$$m_k \leq \varphi(n)2^{(\mu+r)\lg n}2^n(2^{-n})^{2^r} = \varphi(n)\frac{2^{(\mu+r)\lg n}}{2^{(2^r-1)n}}.$$

Asymptotical estimate (7) implies $\lim_{n\to\infty} m_k = 0$.

The proof of the second statement follows from 2, if we show that $\binom{n}{k}\varphi(n)\sqrt{2^{n-k}p^{k2^{k-1}}} = o(\binom{n}{k}2^{n-k}p^{k2^{k-1}})$ for $k \leq \mu - 2$, i.e.

$$\lim_{n\to\infty} \frac{\varphi(n)}{\sqrt{2^{n-k}p^{k2^{k-1}}}} = 0.$$

Since $\varphi(n)$ is arbitrary increasing function, it is sufficient to show $\lim_{n\to\infty} 2^{n-k}p^{k2^{k-1}} = \infty$, for $k \leq \mu - 2$. Let $m_k = 2^{n-k}p^{k2^{k-1}}$. Then, for $k \leq \mu - 2$:

$$m_k \geq 2^{n-\mu+2}p^{(\mu-2)2^{\mu-3}}.$$

Applying (5) we get

$$m_k > 2^{n\mu+2}(2^{-n})^{\frac{\mu-2}{2\mu}}$$
$$m_k > 2^{\frac{n}{2}-\mu+2}.$$

Asymptotical estimate (7) implies $\lim_{n\to\infty} m_k = \infty$. $\square$

Theorem 3 easily follows from upper bound (6) and Lemma 4:

Theorem 3 (nonexistence of big cubes)**.** Any graph $G \in G^n$ does not contain cubes of order $k \geq \lg n - \lg \log_b 2^n - \lg \lg \log_b 2^n + 3$, with probability converging to 1 as $n \to \infty$.

The number of edges of random graph $G \in G^n$ is another quantity we needed to estimate. We denote by h_n a random variable counting edges of G. Trivially $h_n = X_{n,1}$ (each edge is a 1-cube). Thus, the results for cubes in random graph can be used for estimating h_n.

Theorem 4. The following statements hold for any $G \in G^n$, with probability converging to 1 as $n \to \infty$:

$$n2^{n-1}p - n^2\sqrt{2^{n-1}p} < h_n < n2^{n-1}p + n^2\sqrt{2^{n-1}p} \tag{8}$$
$$h_n \sim n2^{n-1}. \tag{9}$$

Proof. Inequalities (8) follow from Theorem 2, for $\varphi(n) = n$ and $k = 1$. Estimate (9) is a corollary of Lemma 4. $\square$

5 CUBES CONTAINING A FIXED VERTEX

Recall that $\deg_k(v, G)$ denotes the number of k-cubes in G containing vertex v.

Lemma 5. Let $0 \leq k \leq k_0$, where $k_0 = \lceil \lg \log_b n + 2 \rceil$. Let $P_{n,k}(v)$ be a probability that for random graph G the following inequality holds:

$$\left| \deg_k(v, G) - \binom{n}{k} p^{k2^{k-1}} \right| \geq \frac{1}{k_0} \binom{n}{k} p^{k2^{k-1}}.$$

Then

$$P_{n,k}(v) \leq \frac{ck_0^5}{n}.$$

Proof. Let $Y_{n,k}$ be a random variable denoting the number of k-cubes in G containing vertex v. It can be easily seen that $\mathrm{E}(Y_{n,k}) = \binom{n}{k} p^{k2^{k-1}}$.

Let η_K be a 0/1 random variable (indicator) attaining value 1, if and only if cube $K \subseteq G$ contains v. Let us express $\mathrm{E}(Y_{n,k}^2)$:

$$\begin{aligned}\mathrm{E}(Y_{n,k}^2) &= \mathrm{E}(\sum_{(K,L)} \eta_K \eta_L) = \sum_{(K,L)} \Pr[\eta_K \eta_L = 1] \\ &= \sum_{K \cap L = \{v\}} \Pr[\eta_K \eta_L = 1] + \sum_{K \cap L \neq \{v\}} \Pr[\eta_K \eta_L = 1],\end{aligned}$$

where the sum is taken over all order pairs (K, L), such that K, L are k-cubes containing v. If $K \cap L = \{v\}$, then $\Pr[\eta_K \eta_L = 1] = p^{k2^k}$. Therefore

$$\sum_{K \cap L = \{v\}} \Pr[\eta_K \eta_L = 1] = \binom{n}{k}\binom{n-k}{k} p^{k2^k} \leq \binom{n}{k}^2 p^{k2^k} = \mathrm{E}^2(Y_{n,k}).$$

If $K \cap L \neq \{v\}$, we get (j denotes order of cube $K \cap L$)

$$\sum_{K \cap L \neq \{v\}} \Pr[\eta_K \eta_L = 1] = \sum_{j=1}^{n} \binom{n}{j}\binom{n-k}{k-j}\binom{n-j}{k-j} p^{k2^k - j2^{j-1}}.$$

The largest summand is the first one ($j = 1$), for $k \leq k_0$. Hence, the sum can be estimated

$$\sum_{K \cap L \neq \{v\}} \Pr[\eta_K \eta_L = 1] \leq k_0 n \binom{n-1}{k-1}^2 p^{k2^k - 1} \leq \frac{ck_0^3}{n}\,\mathrm{E}^2(Y_{n,k}).$$

Putting all these estimates together, the variance of $Y_{n,k}$ can be expressed:

$$\begin{aligned}\mathrm{Var}(Y_{n,k}) &= \mathrm{E}(Y_{n,k}^2) - \mathrm{E}^2(Y_{n,k}) \\ &\leq \mathrm{E}^2(Y_{n,k}) + \frac{ck_0^3}{n}\,\mathrm{E}^2(Y_{n,k}) - \mathrm{E}^2(Y_{n,k}) \\ &= \frac{ck_0^3}{n}\,\mathrm{E}^2(Y_{n,k}).\end{aligned}$$

The Lemma follows from Chebyshev inequality for random variable $Y_{n,k}$ by setting $\varepsilon = \frac{1}{k_0}\,\mathrm{E}(Y_{n,k})$. □

Remark 1. In an unpublished manuscript we have shown that the orders of maximal cubes that cover asymptotically all vertices of a random graph lie in interval

$$(\lg \log_b n - \lg \lg \log_b n, \lg \log_b n + 2).$$

Let $b_k(G)$ be the number of vertices from G such that $|\deg_k(v, G) - \mathrm{E}(Y_{n,k})| \geq \frac{1}{k_0}\,\mathrm{E}(Y_{n,k})$. The expected value of b_k can be upper-bounded by help of Lemma 5:

$$\mathrm{E}(b_k(G)) = \sum_{v \in G} P_{n,k}(v) \leq \frac{ck_0^5}{n} \cdot 2^n.$$

Applying Markov inequality to $b_k(G)$ yields

$$\Pr[b_k(G) \geq k_0 \operatorname{E}(b_k(G))] \leq \frac{1}{k_0}.$$

Since $k_0 \to \infty$ for $n \to \infty$ we get

$$\lim_{n\to\infty} \Pr[b_k(G) \leq \frac{ck_0^6}{n} \cdot 2^n] = 1. \tag{10}$$

This estimate holds for any $k \leq k_0$. Let $\varepsilon_G(k_0)$ be a relative number of those vertices, such that $\deg_{k_0}(v, G) < \binom{m}{k_0} p^{k_0 2^{k_0-1}}(1 - \frac{1}{k_0})$. Applying (10) we get

$$\lim_{n\to\infty} \Pr[\varepsilon_G(k_0) \leq \frac{ck_0^6}{n}] = 1.$$

We restate this result as lemma.

Lemma 6. Let $\varepsilon_G(k_0)$ be a relative number of those vertices, such that $\deg_{k_0}(v, G) < \binom{m}{k_0} p^{k_0 2^{k_0-1}}(1 - \frac{1}{k_0})$. Then with probability converging to 1 as $n \to \infty$, the following inequality holds:

$$\varepsilon_G(k_0) \leq \frac{ck_0^6}{n}.$$

Let us summarize the obtained results – for random graph G with probability converging to 1 as $n \to \infty$:

1. G does not contain cubes of order greater than $\mu \sim \lg n - \lg\lg \frac{1}{p}$,
2. $$X_{n,k} \sim \binom{n}{k} 2^{n-k} p^{k 2^{k-1}}, \text{ for } k \leq \mu - 2,$$
3. $$\varepsilon_G(k_0) \leq \frac{ck_0^6}{n}, \text{ for some constant } c.$$

It follows from our discussion that the length of greedy covering of the random graph by cubes can be estimated by Theorem 1 for the parameters

$$d = \binom{n}{k_0} p^{k_0 2^{k_0-1}} \left(1 - \frac{1}{k_0}\right), \qquad \varepsilon = \varepsilon_G(k_0), \qquad k = k_0$$
$$|V| = 2^n, \qquad |H'| = \binom{n}{k_0} 2^{n-k_0} p^{k_0 2^{k_0-1}}.$$

Simplification leads to the following theorem:

Theorem 5. The length of greedy covering of the random graph by cubes is, with probability converging to 1 as $n \to \infty$, at most

$$\frac{2^n}{\log_b n}\,(1 - o(1)).$$

Acknowledgments

The first author was supported by VEGA grant No. 1/3022/06. The second author was supported by VEGA grant No. 1/3106/06.

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