Computing and Informatics, Vol. 21, 2002, 433–454

INDUCED PARAMETRISATION AND ITS APPLICATIONS IN GEOMETRIC COMPUTATION

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Manuscript received 5 September 2002; revised 27 November 2002

Communicated by Horst Bischhof

Abstract. The paper describes a concept of induced rational parametrisation for curves. Parametrisations of curves are defined in terms of rational parametrisations of simpler or 'primitive' curves. The technique has applications in computer graphics and geometric modeling. A range of examples is given.

Keywords: Computer graphics, visualization, geometric modeling, rational parametrization, geometric computation, computer-aided design, computer-aided geometric design

1 INTRODUCTION

Rendering and geometric algorithms in computer graphics and geometric modeling make extensive use of both the implicit and parametric representations of curves and surfaces. Implicit representation is often preferred for problems such as point classification, whereas parametric representation is usually more convenient for rendering [6, 9].

Rational parametrisation has become the de facto industry standard for parametric representations — the reasons for this include: rational parametrisation requires the evaluation of polynomial functions only, it allows interactive control of the geometry and is complete in the sense that approximation to any tolerance can be achieved and exact rational representation is often possible.

This paper introduces an augmented linear algebra of paths as a means of inducing rational parametrisations of curves and surfaces defined in terms of 'primitive' paths for which rational representations exist. Parametrisations are induced on the circle and other 'standard' curves. The applications presented include the modeling of a rotary engine combustion chamber, the rational representation of classical analytic airfoils and a family of rational alternatives to the Fermat curves and the super-ellipses.

2 CURVES AND PATHS

2.1 Definitions

The C^∞ function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is said to be regular at (x_0, y_0) if $\nabla f(x_0, y_0) \neq 0$. If f is regular everywhere then the set of points $\mathcal{C}_f = \{(x, y) : f(x, y) = 0\}$ defines a regular curve in $\mathbb{R}^2$. The relationship $f(x, y) = 0$ is known as the implicit representation of the curve $\mathcal{C}_f$.

If I is a closed and bounded interval of $\mathbb{R}$, then a regular path in the plane is a C^1 function $p\colon I \to \mathbb{R}^2$ with $p' \neq 0$. The set of paths on I is denoted $\mathcal{P}_I$.

If $p \in \mathcal{P}_I$ is such that $p(t) \in \mathcal{C}_f$ for all $t \in I$ then p is said to be a (local) parametrisation of the curve $\mathcal{C}_f$.

Paths $p \in \mathcal{P}_I$ and $q \in \mathcal{P}_I$ parametrise the same curve (or are equivalent) if and only if there is a C^1 function $\phi : I \to I$ such that $q = p \circ \phi$ and $\phi' \neq 0$ on I. Equivalent paths have identical graphs in $\mathbb{R}^2$.

Similarly a surface may be defined implicitly in terms a suitably regular function $f : \mathbb{R}^3 \to \mathbb{R}$ by $\{(x, y, z) : f(x, y, z) = 0\}$ and (locally) parametrised by a surface 'patch' $s\colon I_1 \times I_2 \to \mathbb{R}^3$.

2.2 Rational Curves

A curve $\mathcal{C}_f$ is said to be rational if it is possible to parametrise it by rational functions; i.e., if there is a path of the form $p = \frac{1}{Q}(P_1, P_2)$, where Q, P_1 and P_2 are polynomial functions, that parametrises $\mathcal{C}_f$.

It can be shown by elementary means that the functional equation

$$f_1(t)^2 + f_2(t)^2 = 1$$

has no non-trivial polynomial solutions for f_1 and f_2. It follows that no exact polynomial representations of the circle $\{(x, y) : x^2 + y^2 - 1 = 0\}$, or any circular arc, exist. However, polynomial solutions of the functional equation

$$\xi_1(t)^2 + \xi_2(t)^2 = \xi_3(t)^2$$

do exist, and their generic form is known [12]. Hence explicit rational parametrisations of circles and circular arcs can be constructed.

However, rational parametrisations do not exist for all curves; for example, it can be shown by elementary means [15] that the Fermat curves

$$\{(x, y) : x^n + y^n - 1 = 0\}$$

are not rational for any $n \geq 3$. For even n the paths are closed.

3 RATIONAL PATHS IN GEOMETRIC MODELING

The interval $I = [0, 1]$ and the Bézier-Bernstein basis $b_{n,k}(t) = \binom{n}{k} t^k (1-t)^{(n-k)}$ are usually preferred in applications. The general rational polynomial path of degree n may then be expressed in the form

$$p_B(t) = \frac{\sum_{k=0}^{n} b_{n,k}(t)\omega_k v_k}{\sum_{k=0}^{n} b_{n,k}(t)\omega_k}$$

for $0 \leq t \leq 1$. The scalars $\omega_0, \ldots, \omega_n$ are called the weights of the path and the vectors $v_0, \ldots, v_n$ are the vertices. The Bézier-Bernstein representation imparts the convex-hull and other desirable properties - provided the weights $\omega_0, \ldots, \omega_n$ are all positive [5]. Figure 1 shows the graph of a typical rational Bézier path of degree 10 with the vertices and 'polygon' displayed.

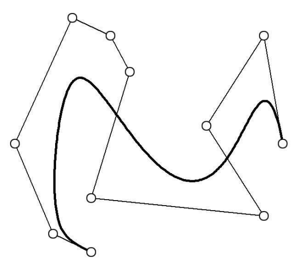


Fig. 1. Rational Bézier path

4 PATH ALGEBRA

4.1 Path Algebra Operations

For any path p we can define a path λp, for $\lambda \in \mathbb{R}$ by $(\lambda p)(t) = \lambda(p(t))$ and the sum $p + q$ of paths p and q in $\mathcal{P}_I$ by $(p + q)(t) = p(t) + q(t)$. Under these operations $\mathcal{P}_I$ is a vector space over $\mathbb{R}$. A binary operator $*$ can be defined on $\mathcal{P}_I$ by

$$p * q = (p_1 q_1 - p_2 q_2, p_1 q_2 + p_2 q_1),$$

where $p = (p_1, p_2)$ and $q = (q_1, q_2)$. The $*$ operator is commutative and corresponds to the multiplication of complex-valued functions defined on a real domain.

The fundamental algebraic properties of $\mathcal{P}_I$, under the operations defined, are summarised in the following proposition.

Proposition 1. Under scalar multiplication, $+$ and $*$, the set $\mathcal{P}_I$ is a commutative (linear) algebra with identity over $\mathbb{C}$.

For the purposes of this paper, the following further algebraic operations on $\mathcal{P}_I$ are required:

1. let C_I denote the set of all C^1 functions $f : I \to \mathbb{R}$. The set C_I is a vector space under addition and scalar multiplication. A (left) multiplication of C_I on $\mathcal{P}_I$ can be defined as follows: for $f \in C_I$ and $p \in \mathcal{P}_I$ we define $fp \in \mathcal{P}_I$ by $(fp)(t) = f(t)p(t)$,
2. let Φ_I denote the set of C^1, onto functions from I to I (not necessarily one-to-one) then a (right) 'multiplication' of Φ_I on $\mathcal{P}_I$ can be defined by

   $$\circ : (p, \phi) \to p \circ \phi,$$

   where $p \in \mathcal{P}_I$ and $\phi \in \Phi_I$,
3. the affine group A_2 of $\mathbb{R}^2$ acts as a (left) transformation group on $\mathcal{P}_I$; define for each $g \in A_2$ and $p \in \mathcal{P}_I$ the path gp by $(gp)(t) = Ap(t) + a$, where the pair $g = (A, a)$ is an element A_2. Here A is an invertible linear transformation of $\mathbb{R}^2$ and $a \in \mathbb{R}^2$.

Definition 1. We call the linear algebra $\mathcal{P}_I$, together with the associated (left) multiplications of C_I and A_2 and the (right) multiplication of Φ_I, the algebra of planar paths on I.

4.2 Fundamental Properties

For $\phi' \neq 0$ on I, right-multiplication by ϕ produces an equivalent parametrisation of the same curve. Later in the paper, the role of right-multipliers in the process of inducing rational representations will be discussed and illustrated; the following properties are fundamental to this:

(i) for some non-rational paths $p \in \mathcal{P}_I$, a right-multiplier $\phi \in \Phi_I$ can be found such that $p \circ \phi$ is rational, and

(ii) the homomorphic property of right-multiplication; i.e., if

$$e(p, ..., f, ..., A, ..., \psi, ...) \in \mathcal{P}_I$$

is a path algebra expression, where $p \in \mathcal{P}_I$, $f \in C_I$, $A \in A_2$ and $\psi \in \Phi_I$ then for all $\phi \in \Phi_I$ we have

$$e(p, ..., f, ..., A, ..., \psi ...) \circ \phi = e(p \circ \phi, ..., f \circ \phi, ..., A, ..., \psi \circ \phi, ...).$$

If $\phi' \neq 0$ on I then the paths $e(p, ..., f, ..., A, ..., \psi)$ and $e(p, ..., f, ..., A, ..., \psi) \circ \phi$ are equivalent.

If $e(p, q, f, A) = fp * q + A(p * p)$ then using the homomorphic property of right multiplication and the definition of functional composition, we have:

$$\begin{aligned} e(p, q, f, h, A) \circ \phi &= (fp * q + A(p * p)) \circ \phi \\ &= (f \circ \phi)(p \circ \phi) * (q \circ \phi) + A((p \circ \phi) * (p \circ \phi)) \\ &= e(p \circ \phi, q \circ \phi, f \circ \phi, A). \end{aligned}$$

4.3 The Denotation of Path Algebra Expressions

Care is required in the denotation of expressions in the extended algebra as, for example, neither the $*$ operator nor the right multiplication of C_I associate with the left multiplication of Φ_I on $\mathcal{P}_I$; i.e., although the algebraic expressions such as

$$f(p \circ \phi) \quad \text{and} \quad (fp) \circ \phi$$

are both valid and determine well-defined paths for all f and ϕ, they are not equal. Similarly

$$(p^2) \circ \phi \quad \text{and} \quad p * (p \circ \phi).$$

Brackets, or binary tree structures, must therefore be used in such expressions to convey meaning. The (left) affine action associates with $\circ$ but not with $*$; i.e., we have

$$(Ap) \circ \phi = A(p \circ \phi)$$

for all affine A and $\phi \in \Phi_I$, but

$$A(p * p) \neq (Ap) * p.$$

In the remainder of the paper we refer to $\mathcal{P}_I$, with the operations defined, as a 'Constructive Path Algebra (or CPA)' and unambiguous constructions in $\mathcal{P}_I$ as CPA expressions. A binary tree for the CPA expression $A(p * p) + f(p \circ \phi)$ is shown in Figure 2.

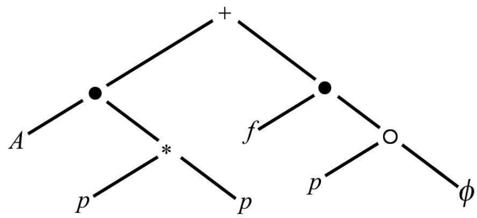


Fig. 2. Binary tree for CPA expression

4.4 Subalgebras Generated by Particular Paths

If $p = (p_1, p_2) \in \mathcal{P}_I$ we say a path $q \in \mathcal{P}_I$ is rationally related to p if q has the form

$$q = \frac{1}{Q(p_1, p_2)}(P_1(p_1, p_2), P_2(p_1, p_2))$$

where Q, P_1 and (P_2) are polynomial functions in two variables. We denote by $\{p\}_r$ the set of all paths rationally related to p. It can be shown that $\{p\}_r$ is a subalgebra of $\mathcal{P}_I$ generated from p by restricting the left multipliers to the form $H \circ p$ where $H = \frac{K}{L}$ and K and L are polynomial functions of two variables and only allowing trivial right multiplication - i.e., by $\phi(t) = t$ [1, 2].

5 INDUCED PARAMETRISATIONS

5.1 The Principle of Induced Parametrisation

Path algebra may be used to express a path q as a function $e(p)$ of a 'primitive' path p for which rational parametrisations are known - i.e., $p \circ \phi$ is a rational parametrisation of p for some right-multiplier ϕ. If $e(p) \in \{p\}_r$ it follows that a rational parametrisation may then be 'induced' on q by applying ϕ to $e(p)$; i.e., for all $e(p) \in \{p\}_r$,

$$e(p) \circ \phi = e(p \circ \phi) \quad \text{is a rational parametrisation of } e(p).$$

We note that the right multiplier ϕ will be trivial, i.e., $\phi(t) = t$ for all $t \in I$, in the case where the parametrisation of the inducing primitive, p, is apriori rational.

The remainder of this section shows how a known series of rational parametrisations of the circle can, within the context of path algebra, be induced by suitably chosen straight line primitives and how rational parametrisations of the non-circular conic sections can be induced by the circle.

5.2 Induced Rational Parametrisations of the Circle

Consider, for $n \geq 1$ the straight line segment l_n between the points $(1,0)$ and $(\cos\frac{\pi}{n}, \sin\frac{\pi}{n})$. Parametrise l_n on $I = [0,1]$ by

$$L_n(t) = (1-t)(1,0) + t\left(\cos\frac{\pi}{n}, \sin\frac{\pi}{n}\right),$$

this is a degree one, and hence a rational, parametrisation. Denoting the Euclidean norm on $\mathbb{R}^2$ by $\|\cdot\|$, we construct within the path algebra $\{L_n\}_r$ the path $e(L_n)$, where

$$e(L_n) = \frac{1}{\|L_n\|^{2n}} L_n^{2n}$$

for $L_n^{2n} = L_n * L_n * \cdots * L_n$. It follows that $\|e(L_n)(t)\| = 1$ for all $t \in [0,1]$ and that $e(L_n)$ is a degree $2n$ rational parametrisation of the complete circle $p(t) = (\cos(2\pi t), \sin(2\pi t))$ on $0 \leq t \leq 1$. The parametrisation $e(L_n)$ of the circle is induced by the parametrisation L_n of the line l_n. This is an example for which the right multiplier ϕ is trivial, as the parametrisation L_n of the inducing path l_n is rational.

The series $e(L_n)$ of even degree rational parametrisations constructed above is equivalent to that described in [16].

The following proposition is easy to prove.

Proposition 2. For $n \geq 3$ the weights of the induced rational parametrisation $e(L_n)$ of the circle are positive.

As each of the paths $e(L_n)$ is equivalent to $p(t) = (\cos(2\pi t), \sin(2\pi t))$, it follows that there exist right multipliers, $\phi_n : [0,1] \to [0,1]$, such that on $[0,1]$ we have

$$p \circ \phi_n = e(L_n),$$

i.e. ϕ_n 'transforms' p from transcendental to rational form. The explicit form of ϕ_n can be shown to be

$$\phi_n(t) = \frac{n}{\pi}\tan^{-1}\left(\frac{2t-1}{\gamma_n}\right),$$

where

$$\gamma_n = \frac{1+\cos\frac{\pi}{n}}{4\sin\frac{\pi}{n}}.$$

It is easy to show that as n increases the rational parametrisations $p \circ \phi_n$ approach the 'ideal' (i.e., arc length) parametrisation of the circle.

5.3 Induced Rational Parametrisations of Circular Arcs

Positive weight, rational parametrisations of partial circles are also useful in applications and can be constructed using the same technique. For example, from

$$L_4(t) = (1-t)(1,0) + t\left(\cos\frac{\pi}{3}, \sin\frac{\pi}{3}\right)$$

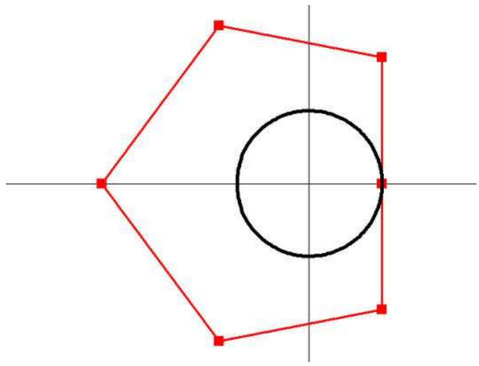


Fig. 3. Induced degree 6 rational parametrisation of the circle

we can construct

$$e(L_4) = \frac{1}{\|L_4\|^2} L_4^2,$$

which is a positive weight, degree 2, rational parametrisation of the unit circular arc subtending 0 to $\frac{2\pi}{3}$ on the parametric interval $0 \leq t \leq 1$.

For the general degree-one function

$$L(t) = (1-t)(x_0, y_0) + t(x_1, y_1),$$

where $t \in I$, the corresponding quadratic rational function $\frac{L^2}{\|L\|^2}$ has weights:

$$\omega_0 = x_0^2 + y_0^2, \;\; \omega_1 = x_0 x_1 + y_0 y_1, \;\; \omega_2 = x_1^2 + y_1^2$$

and vertices, defined for $\omega_i \neq 0$, by:

$$v_0 = \frac{(x_0^2 - y_0^2, 2x_0 y_0)}{\omega_0}, v_1 = \frac{(x_0 x_1 - y_0 y_1, x_0 y_1 + x_1 y_0)}{\omega_1}, v_2 = \frac{(x_1^2 - y_1^2, 2x_1 y_1)}{\omega_2}.$$

Hence if

$$(x_0, y_0) \cdot (x_1, y_1) > 0$$

the weights are positive.

5.4 Induced Rational Parametrisations of the Non-Circular Conic Sections

In the previous section it was shown how rational parametrisations of the circle can be induced from rational (actually polynomial) parametrisations of straight line segments. The circle may also be used as a primitive for inducing parametrisations of curves. For example the ellipse, parabola and hyperbola may be written, in a suitable polar coordinate system, as a function of the circle p. Writing $p = (p_1, p_2)$ where $p_1(t) = \cos(2\pi t)$ and $p_2(t) = \sin(2\pi t)$ the hyperbola with implicit form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0,$$

shown in Figure 4, has the parametric form

$$\frac{a(\epsilon^2 - 1)}{1 - \epsilon\cos(\theta)}\,(\cos(\theta), \sin(\theta)) + (a\epsilon, 0) \quad \text{for} \;\; 0 \le \theta \le 2\pi.$$

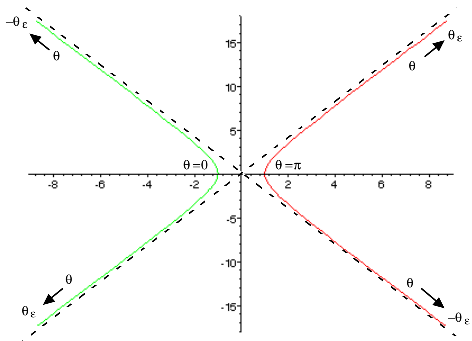


Fig. 4. Graph of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$

The left branch, h, has the parametric domain $-\theta_\epsilon < \theta < \theta_\epsilon$ where $\epsilon = \frac{(a^2+b^2)^{\frac{1}{2}}}{a}$ and $\theta_\epsilon = \cos^{-1}\left(\frac{1}{\epsilon}\right)$.

Hence h can be written as a path algebra function of the circular arc primitive $p_\epsilon(\theta) = (\cos\theta, \sin\theta)$, $-\theta_\epsilon \le \theta \le \theta_\epsilon$ as

$$e_h(p_\epsilon) = \frac{a(\epsilon^2 - 1)}{1 - \epsilon p_{\epsilon 1}} p_\epsilon + (a\epsilon, 0)$$

where $p_\epsilon = (p_{\epsilon 1}, p_{\epsilon 2})$. This representation provides a means of inducing rational parametrisations of h from rational parametrisations of p_ϵ.

A quadratic rational parametrisation of p_ϵ can be built in path algebra from the degree one path

$$L_\epsilon(t) = (1 - t)(1, 0) + t\,(\cos\theta_\epsilon, \sin\theta_\epsilon)$$

and the rotation transformation $R_{-\theta_\epsilon}$ through $-\theta_\epsilon$; we define it as a path algebra function of L_ϵ by

$$e(L_\epsilon) = \frac{R_{-\theta_\epsilon}(L_\epsilon * L_\epsilon)}{\|L_\epsilon\|^2}$$

and, writing $e(L_\epsilon) = \frac{1}{Q}(P_1, P_2)$, we have

$$\begin{aligned} P_1(t) &= (1-t)^2 \cos\theta_\epsilon + 2(1-t)t + t^2 \cos\theta_\epsilon, \\ P_2(t) &= -(1-t)^2 \sin\theta_\epsilon + t^2 \sin\theta_\epsilon, \\ Q(t) &= (1-t)^2 + 2(1-t)t\ \cos\theta_\epsilon + t^2. \end{aligned}$$

The rational quadratic arc $e(L_\epsilon)$, which is defined on $[0,1]$ and shown in Figure 5, satisfies

$$p_\epsilon \circ \phi_\epsilon = e(L_\epsilon)$$

for some right-multiplier $\phi_\epsilon : [0,1] \to [-\theta_\epsilon, \theta_\epsilon]$.

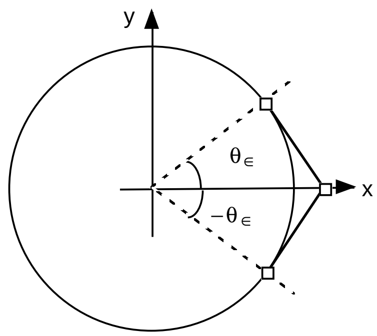


Fig. 5. Bézier arc $p_\epsilon \circ \phi_\epsilon$ for inducing a rational parametrisation of h

Applying ϕ_ϵ to $e_h(p_\epsilon)$ gives

$$\begin{aligned} e_h(p_\epsilon) \circ \phi_\epsilon &= e_h(p_\epsilon \circ \phi_\epsilon) \\ &= \left(\frac{a(\epsilon^2 - 1)}{1 - \epsilon p_{\epsilon,1} \circ \phi_\epsilon} \right) p \circ \phi_\epsilon + (a\epsilon, 0) \\ &= \frac{a(\epsilon^2-1)(P_1, P_2) + (Q - \epsilon P_1)(a\epsilon, 0)}{Q - \epsilon P_1}. \end{aligned}$$

Using the relationships $\cos\theta_\epsilon = \frac{a}{c}$, and $\sin\theta_\epsilon = \frac{b}{c}$, where $c = \sqrt{a^2+b^2}$, and the expressions for P_1, P_2 and Q derived above we obtain the following quadratic rational parametrisation of h on the interval $0 \le t \le 1$:

$$e_h(p_\epsilon \circ \phi_\epsilon)(t) = \frac{(1-t)^2(-a,\ b) + 2t(1-t)(0,\ 0) + t^2(-a,\ -b)}{2(1-t)t}.$$

Whilst this not a standard rational Bézier form, due to the zero weights associated with the basis polynomials $(1-t)^2$ and t^2, this rational representation of h is required for one of the applications discussed later in the paper.

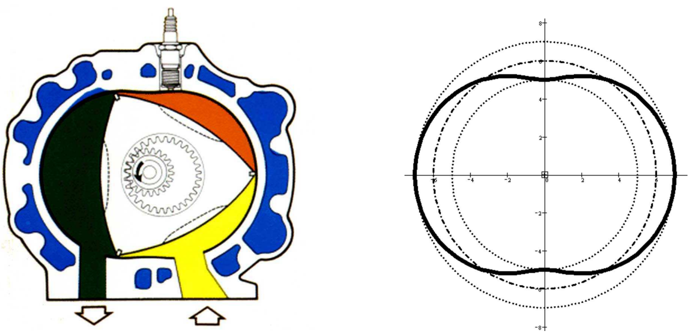
6 APPLICATIONS IN GEOMETRIC MODELING

6.1 A Rotary Engine Combustion Chamber

The combustion chamber of the Wankel rotary engine has an epitrochoidal cross section [7] of the form

$$p_w(t) = (6\cos(2\pi t) + \cos(6\pi t), 6\sin(2\pi t) + \sin(6\pi t)); \quad \text{for} \quad 0 \le t \le .$$

The internal components of the engine and the cross section of the combustion chamber are shown in Figure 6.



a) b)

Fig. 6. The Wankel rotary engine
a) The internal components
b) Chamber cross-section

The path p_w can be written in the path algebra of the circle primitive $p(t) = (\cos 2\pi t, \sin 2\pi t)$ as

$$e_w(p) = 6p + p * p * p.$$

6.1.1 Global Parametrisations of p_w

Now p is such that $p \circ \phi_n = e(L_n)$, hence ϕ_n can be used to transform p_w to a rational form on $[0, 1]$; we have:

$$\begin{aligned} e_w(p) \circ \phi_n &= (6p + p * p * p) \circ \phi_n \\ &= 6(p \circ \phi_n) + (p \circ \phi_n) * (p \circ \phi_n) * (p \circ \phi_n) \\ &= 6e(L_n) + e(L_n) * e(L_n) * e(L_n) \\ &= \frac{6{L_n}^{2n} \|L_n\|^{4n} + {L_n}^{6n}}{\|L_n\|^{6n}} \end{aligned}$$

which provides, for $n \geq 3$, a positive-weight, degree $6n$, rational representations of the epitrochoid p_w on the interval $[0, 1]$.

The vertices $v_i = (x_i, y_i)$ and associated weights ω_i for $p_w \circ \phi_n$ can be calculated by standard basis-conversion methods. The degree 18 path, corresponding to $n = 3$ and its Bézier polygon are shown in Figure 7. This example demonstrates that global rational representations tend to be of high degree and can give rise to control polygons having a generally non-intuitive shape.

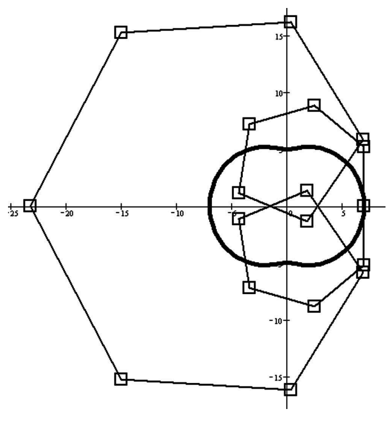


Fig. 7. The single-segment, degree 18 path with Bézier vertices shown

A degree 6 global parametrisation of p_w is possible and this can be induced using the well-known quadratic parametrisation of the circle defined by $\frac{1}{1+t^2}(1 - t^2, 2t)$ on $(-\infty, \infty)$. However, this parametrisation of p_w has a number of disadvantages from the point of view of geometric computation; for example (i) the parametric domain is not finite, and (ii) zero weights occur. A degree 15 parametrisation can be induced from Chou's quintic parametrisation of the circle [3], and this is probably the lowest possible degree parametrisation of p_w without these drawbacks.

6.1.2 Local Parametrisation of p_w

By exploiting the geometric symmetry of the epitrochoid, a lower-degree rational representation of p_w may be constructed using the path algebra. Using L_4 we induce the positive-weight rational parametrisation $e(L_4)$ of the $\frac{1}{4}$-circle in the first quadrant. Applying the associated rational-conversion function ϕ to p_w we obtain

$$\begin{aligned} e_w(p) \circ \phi &= 6(p \circ \phi) + (p \circ \phi)^3 \\ &= 6e(L_4) + e(L_4) * e(L_4) * e(L_4), \end{aligned}$$

which is a degree 6 positive-weight rational representation of the $\frac{1}{4}$-epitrochoid - as shown in Figure 8. For this $\frac{1}{4}$-path solution, the control polygon more clearly reflects the shape of the path. Sánchez-Reyes [17] has obtained a similar parametrisation of this $\frac{1}{4}$-path from a different perspective, however the methods used appear to preclude the construction of global parametrisations.

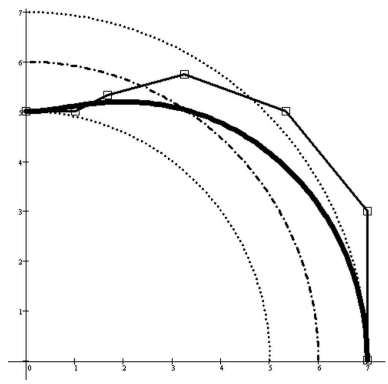


Fig. 8. The $\frac{1}{4}$-path of degree 6 with Bézier vertices shown

6.2 Joukowski Airfoils

The classical definition of the Joukowski airfoil takes the form [13]

$$J(z) = \frac{1}{2}\left(z + \frac{k^2}{z}\right),$$

where z is constrained to a circle C in the complex plane passing through the point $z = -1$ and such that the point $z = 1$ lies inside C. The circle C is related to the unit circle p by an affine transformation, written g, comprising a combination of scale and translation transformations such that the conditions on C are satisfied.

The function J can be written

$$J(z) = \frac{1}{2}\left(z + \frac{k^2\bar{z}}{\|z\|^2}\right)$$

from which it follows that the Joukowski profiles have the CPA form

$$e_J(p) = \frac{1}{2}\left(gp + \frac{k^2}{\|gp\|^2}Agp\right)$$

in the circle algebra $\{p\}_r$. Here A is the affine transformation $A(x,y) = (x,-y)$ and the transformation g of $\mathbb{R}^2$ has the form $g\xi = \lambda\xi + a$, for $\xi \in \mathbb{R}^2$, $\lambda > 0$ and $a \in \mathbb{R}^2$. Hence

$$e_J(p) = \frac{1}{2}\left(\lambda p + a + \frac{k^2 A(\lambda p + a)}{2\lambda p \cdot a + \lambda^2 + \|a\|^2}\right)$$

and, applying the right-multiplier ϕ_n, we obtain

$$\begin{aligned} e_J(p)\circ\phi_n &= e_J(p\circ\phi_n) \\ &= \frac{1}{2}\left(\lambda e(L_n) + a + \frac{k^2 A(\lambda e(L_n) + a)}{2\lambda e(L_n)\cdot a + \lambda^2 + \|a\|^2}\right). \end{aligned}$$

Substituting $e(L_n) = \frac{L_n^{2n}}{\|L_n\|^{2n}}$ gives the explicit form a degree $4n$, exact, positive weight (for $n \geq 3$), rational representation of the Joukowski profile $e_J(p)$ on the interval $[0,1]$. Figure 9 shows the profile, together with its Bézier vertices and the geometric separation of points for equal parametric separation, for the circle corresponding to $\lambda = 1.13344855$, $a = (0.125, 0.13813263)$ and for $k = 1$ and $n = 3$.

That the weights of $e_J(p)$ are positive for all $n \geq 3$ is a consequence of the following proposition.

Proposition 3. If $p\circ\phi$ is a positive-weight rational parametrisation of the unit circle then $e_J(p)\circ\phi$ is a positive-weight rational parametrisation of the Joukowski airfoil.

Proof. By hypothesis

$$p\circ\phi = \frac{(X,Y)}{Q}$$

for some polynomials X, Y, Q where $Q(t) = \sum_i \binom{n}{i}(t^i)(1-t)^{n-i}\omega_i$ with $\omega_i > 0$ and $0 \leq t \leq 1$. Since $Q \geq 0$ on $[0,1]$ it follows from $\|\frac{(X,Y)}{Q}\| = 1$, or equivalently $\|(X,Y)\| = |Q|$, that $\|(X,Y)\| = Q$ on $[0,1]$. We therefore have, denoting the angle

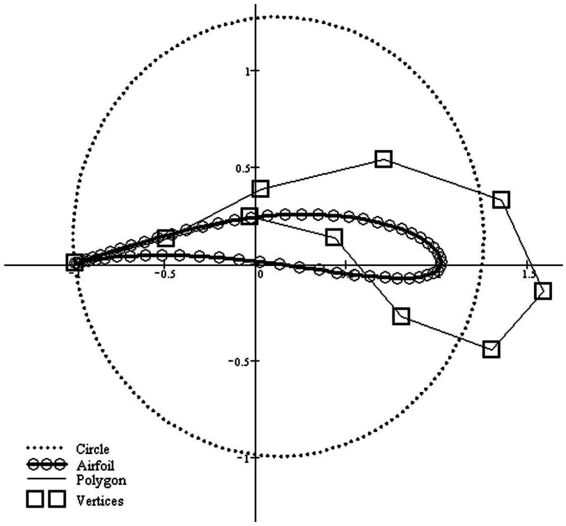


Fig. 9. Joukowski airfoil showing defining circle and Bézier vertices

between (X, Y) and a as δ,

$$\begin{aligned}
e_J(p) \circ \phi &= \frac{1}{2}\left(\lambda p \circ \phi + a + \frac{k^2 A(\lambda p \circ \phi + a)}{2\lambda(p \circ \phi) \cdot a + \lambda^2 + \|a\|^2}\right) \\
&= \frac{1}{2}\left(\lambda \frac{(X,Y)}{Q} + a + \frac{k^2 Q A(\lambda(X,Y) + a)}{2\lambda(X,Y) \cdot a + Q(\lambda^2 + \|a\|^2)}\right) \\
&= \frac{1}{2}\left(\lambda \frac{(X,Y)}{Q} + a + \frac{k^2 Q A(\lambda(X,Y) + a)}{Q(2\lambda\|a\| \cos\delta + \lambda^2 + \|a\|^2)}\right) \\
&= \frac{1}{2}\left(\lambda \frac{(X,Y)}{Q} + a + \frac{k^2 Q A(\lambda(X,Y) + a)}{Q((\lambda + \|a\| \cos\delta)^2 + \|a\|^2(1 - \cos^2\delta))}\right).
\end{aligned}$$

□

Writing $\mu = (\lambda + \|a\| \cos\delta)^2 + \|a\|^2(1 - \cos^2\delta)$ it is clear that $\mu > 0$ and, as Q has positive weights, it follows that the denominator $2\mu Q$ of $e_J(p) \circ \phi$ has positive weights. Hence result.

Recent work with Joukowski foils includes that of [10, 11] and [19].

6.3 Hyperbola Airfoils

The inverse of a path q, with respect to a circle C of radius r and centre $\xi = (\xi_x, \xi_y)$, can be written

$$q^* = \xi + r^2 \frac{(q - \xi)}{\|q - \xi\|^2}.$$

A family of airfoils can be generated by inverting a branch of a hyperbola with respect to a circle having centre at, or near to, the focus of the other branch [14]. In the notation of Section 5.4 it follows that the inverse, h^*, of h with respect to C can be written in path algebra form as

$$h^* = \xi + r^2 \frac{(h-\xi)}{\|h-\xi\|^2}.$$

Earlier it was shown that h can be expressed as a path algebra function $e_h(p_\epsilon)$ in $\{p_\epsilon\}_r$; it follows that h^* can be written as

$$e_{h^*}(p_\epsilon) = \xi + r^2 \frac{(e_h(p_\epsilon)-\xi)}{\|e_h(p_\epsilon)-\xi\|^2}.$$

Applying the right multiplier ϕ_ϵ gives

$$e_{h^*}(p_\epsilon) \circ \phi_\epsilon = \xi + r^2 \frac{(e_h(p_\epsilon)\circ\phi_\epsilon-\xi)}{\|e_h(p_\epsilon)\circ\phi_\epsilon-\xi\|^2}.$$

Writing the rational function $e_h(p_\epsilon)\circ\phi_\epsilon$ as $\frac{H}{Q_h}$, computed explicitly in Section 5.4, yields

$$e_{h^*}(p_\epsilon) \circ \phi_\epsilon = \frac{\xi\|H-Q_h\ \xi\|^2 + r^2(H-Q_h\ \xi)Q_h}{\|H-Q_h\ \xi\|^2},$$

which is a quartic rational parametrisation of the inverted path h^*.

The Bézier vertices $v_i = (x_i, y_i)$ and the normalised weights ω_i of the foil are given, as functions of the hyperbola and the inverting circle, in the following table:

i :	x_i	y_i	ω_i
0 :	ξ_x	ξ_y	1
1 :	$\frac{2a\xi_x^2-2b\xi_x\xi_y-ar^2}{2(a\xi_x-b\xi_y)}$	$\frac{2a\xi_x\xi_y+br^2-2b\xi_y^2}{2(a\xi_x-b\xi_y)}$	$\frac{a\xi_x-b\xi_y}{a^2+b^2}$
2 :	$\frac{\xi_x(a^2-b^2+2(\xi_x^2+\xi_y^2)-2r^2)}{a^2-b^2+2(\xi_x^2+\xi_y^2)}$	$\frac{\xi_y(a^2-b^2+2(\xi_x^2+\xi_y^2)-2r^2)}{a^2-b^2+2(\xi_x^2+\xi_y^2)}$	$\frac{a^2-b^2+2(\xi_x^2+\xi_y^2)}{3(a^2+b^2)}$
3 :	$\frac{2a\xi_x^2+2b\xi_x\xi_y-ar^2}{2(a\xi_x-b\xi_y)}$	$\frac{2a\xi_x\xi_y-br^2+2b\xi_y^2}{2(a\xi_x-b\xi_y)}$	$\frac{a\xi_x+b\xi_y}{a^2+b^2}$
4 :	ξ_x	ξ_y	1.

Table 1. Vertex and weight data for the hyperbola foil h^*

Hence a rational parametrisation of the circular arc p_ϵ induces a rational parametrisation of the left branch h of a hyperbola, which in turn induces a rational parametrisation of the hyperbola foil via the path algebra expression $e_{h^*}(p_\epsilon)$. These general formulae for the vertices and weights, contained in the table, enable the foils to be constructed directly from specified values of a, b, ξ and r — obviating the requirement to invoke the inversion procedure explicitly. Figure 10 shows the graph of the airfoil h^*, the Bézier polygon of the induced quartic parametrisation, the hyperbola h, the inverting circle and corresponding points on h and h^*.

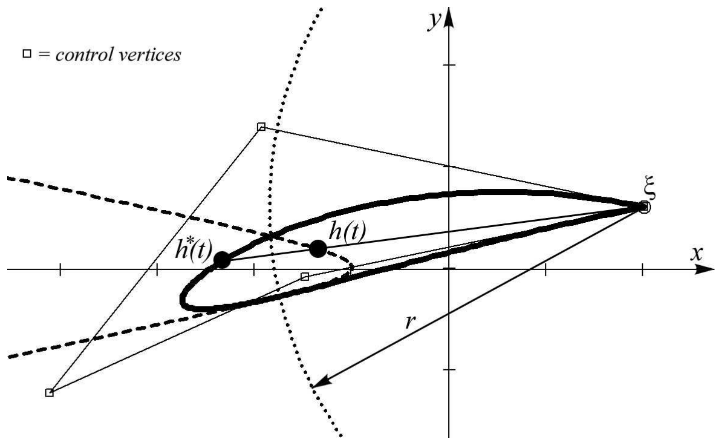


Fig. 10. The induced quartic rational parametrisation of h^*

6.4 Rational Alternatives to the Fermat Curves and the Super-Ellipses

6.4.1 The Fermat Curves

For even values of n the curves are closed and bounded and are of increasing 'fullness' as $n \to \infty$, as shown in Figure 11.

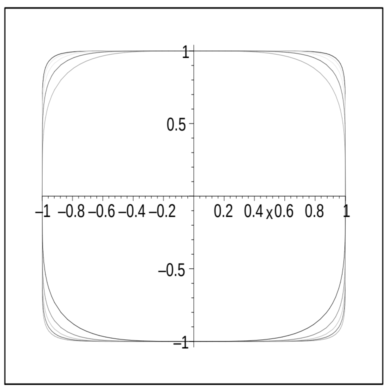


Fig. 11. The Fermat curves for $n = 4, 6, 8, 10, 12$

For odd values of n a set of closed 'Fermat' curves may be defined by the implicit equations

$$|x|^n + |y|^n - 1 = 0.$$

The graphs of these curves are similar to the even integer cases and are of class C^{n-1}.

6.4.2 The Super-Ellipses

The super-ellipses are more general than the Fermat curves and have the implicit forms

$$|\frac{x}{a}|^k + |\frac{y}{b}|^k - 1 = 0$$

for $a, b \in \mathbb{R} \setminus 0$ and where $k \in \mathbb{R}$ with $k > 2$.

If k is rational then it can be shown [18] that the functional equation $x(t)^k + y(t)^k = 1$ has rational solutions for x and y if and only if $k = 2/q$ for $q \in \mathbb{Z}$ and $q \geq 1$.

It follows, for example, that no rational parametrisations of the super-ellipses can exist.

6.4.3 Alternative Paths with Global Rational Parametrisations

The Fermat and super-ellipse curves are primitives for geometric modeling [8]. The Piet Hein super-ellipse corresponds to the choice $k = 5/2$ and has been applied in road design and architecture [20].

The family of curves defined by

$$p_{F,n}(\theta) = \frac{1}{n-1}\left[n(\cos 2\pi t, \sin 2\pi t) - (\cos^n 2\pi t, \sin^n 2\pi t)\right]; \quad \text{for } 0 \leq t \leq 1,$$

for integer $n \geq 3$ have graphs similar to those of the Fermat curves - as shown in Figure 12; however, rational parametrisations of $p_{F,n}$ can be constructed.

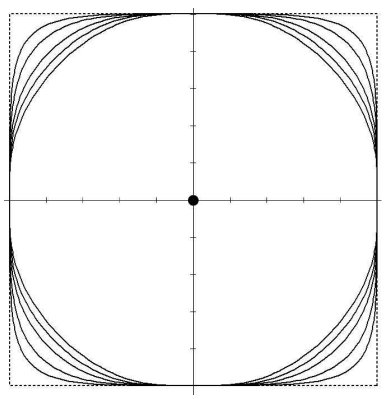
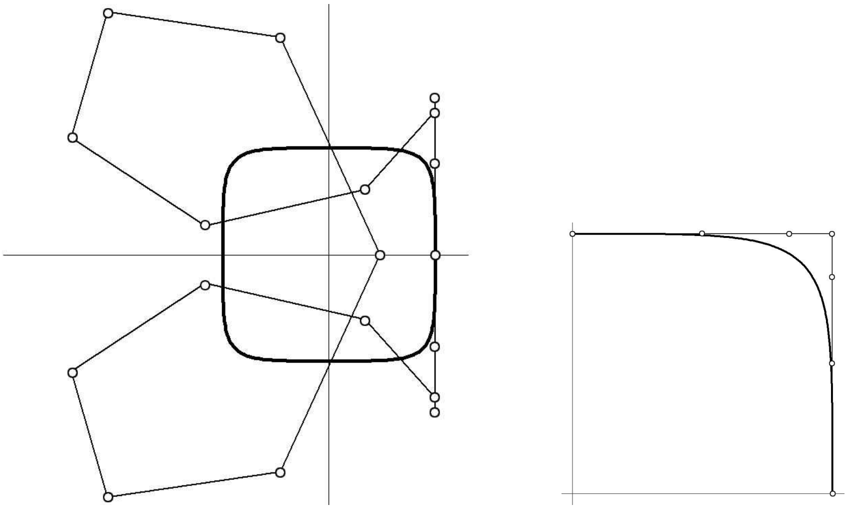


Fig. 12. A family of global rational curves

In path algebra notation we have

$$e_{F,n}(p) = \frac{1}{n-1}\left[np - \frac{1}{2}\left(p^n + A(p^n),\ p^n - A(p^n)\right)\right] \quad \text{on } 0 \leq t \leq 1,$$

where $p(t) = (\cos(2\pi t), \sin(2\pi t))$. The global rational parametrisation, $e_{F,n}(p) \circ \phi_3$, of $p_{F,n}$ is therefore of degree $6n$. The degree 18 path $e_{F,3}(p) \circ \phi_3$ and the $\frac{1}{4}$-path local rational parametrisation $e_{F,3}(p) \circ \phi$ of degree 6 is shown in Figure 13. A similar family of rational paths may be defined as alternatives to the super-ellipses.



a) b)

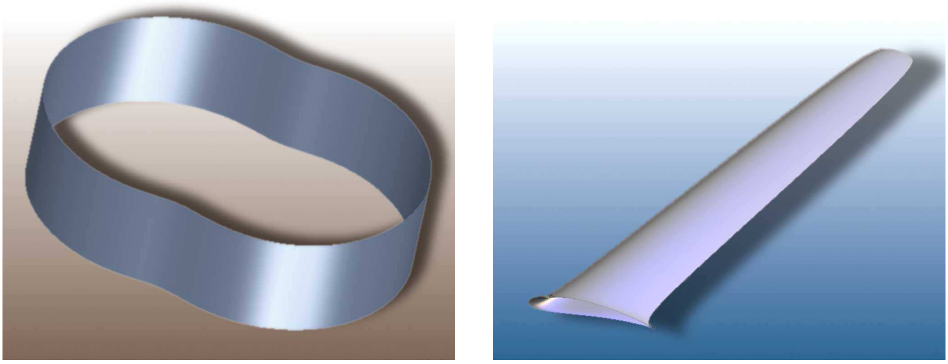
Fig. 13. Rational parametrisations of $p_{F,3}$
a) Global degree 18 path
b) Local degree 6 path

7 SUMMARY AND CONCLUSIONS

The paper has described an elegant induction procedure for the explicit determination of rational parametrisations of many curves and surfaces. The representations are exact and do not require the direct input of vertex or weight information. The examples given have shown how many paths, specified in terms of transcendental functions, may be converted to a rational form compatible with most current computer graphics and geometric computation systems. Using traditional sweeping, lofting and scaling techniques enables exact rational surface patches to be constructed from the rational profiles constructed in the paper. Some examples are shown in Figure 15; the engine chamber wall is defined by the rational patch $s_w : [0,1] \times [0,1] \to \mathbb{R}^3$, where $s_w(t,u) = (e_w \circ \phi_3(t), h\,u)$ and h is the height of the wall, and the wing is constructed by scaling the rational airfoil profile along the rational longitudinal path shown in Figure 14.

The paper concludes by considering alternatives to the Fermat paths - for which it is known that rational parametrisations do not exist. The alternatives suggested

Fig. 14. Degree 7 rational longitudinal path for construction of Joukowski surface



a) b)

Fig. 15. Sample rational surfaces rendered from the induced parametrisations
a) Rotary engine chamber wall
b) Joukowski wing

have global rational parametrisations, are intrinsically C^∞ with C^∞ parametrisations.

REFERENCES

[1] Bez, H. E.—Wetzel, T. J.: A Constructive Path Algebra and its Application to Curve and Surface Modeling. Internal research report 1055, Department of Computer Science, Loughborough University, 2001.

[2] Bez, H. E.—Wetzel, T. J.: Constructive Path Algebra — a Tool for Design, Parametrisation and Visualisation. Proceedings of Eurographics UK 2002, IEEE Computer Society Press, pp. 97–104.

[3] Chou, J. J.: Higher Order Bézier Circles. Computer-Aided Design, Vol. 27, 1995, pp. 303–309.

[4] Farin, G.: Curves and Surfaces for Computer Aided Geometric Design — a Practical Guide. Second ed., Academic Press, Inc., San Diego, 1990.

[5] Farin, G.: NURBS — from Projective Geometry to Practical Use. A.K. Peters Ltd., 1999.

[6] Farouki, R.—Hinds, J. K.: A Hierarchy of Geometric Forms. IEEE Computer Graphics and Applications, Vol. 5, 1985, No. 5, pp. 51–78.

[7] FAROUKI, R. T.—RAMPERSAD, J.: Cycles Upon Cycles: an Anecdotal History of Higher Curves in Science and Engineering. Mathematical Methods for Curves and Surfaces II, pp. 95–116, 1998.

[8] FAUX, I. D.—PRATT, M. J.: Computational Geometry for Design and Manufacture. Second ed., Ellis Horwood, Ltd., Chichester, 1983.

[9] HOSCHEK, J.—LASSER, D.: Fundamentals of Computer Aided Geometric Design. A.K. Peters, 1993.

[10] HUANG, M. K.—CHOW, C. Y.: Trapping of a Free Vortex by Joukowski Airfoils. AIAA Journal, Vol. 20, 1982, No. 3, pp. 292–298.

[11] KLEIN, M.—SOBIECZKY, H.: Sensitivity of Aerodynamic Optimization to Parametrised Target Functions. Proc. Int. Symp. on Inverse Problems (ISIP) in Engineering Mechanics, Elsevier, 2001.

[12] KUBOTA, K. K.: Pythagorean Triples in Unique Factorization Domains. American Mathematical Monthly 79, pp. 503–505, 1972.

[13] PHILLIPS, E. G.: Functions of a Complex Variable. University Mathematical Texts No. 7, Interscience Publishers Inc., New York, 1940.

[14] RIEGELS, F. W.: Aerofoil Sections: Results from Wind-Tunnel Investigations. Theoretical foundations, Butterworth, 1961.

[15] ROE, J.: Elementary Geometry, Oxford University Press, 1993.

[16] SÁNCHEZ-REYES, J.: Higher Order Bézier Circles. Computer-Aided Design, Vol. 31, 1997, No. 12, pp. 747–750.

[17] SÁNCHEZ-REYES, J.: Bézier Representation of Epitrochoids and Hypotrochoids. Computer-Aided Design, Vol. 31, 1999, No. 12, pp. 747–750.

[18] SHANKS, D.: Solved and Unsolved Problems in Number Theory. Chelsea Publishing, New York, 1985.

[19] STRUMOLO, G. S.: ISDP: Inverse Shape Design Procedure. IMA seminars on industrial problems, 1999.

[20] VILLECO, R.: The Infinitely Expandable Future of Air Structures. Architectural Forum, September 1970.

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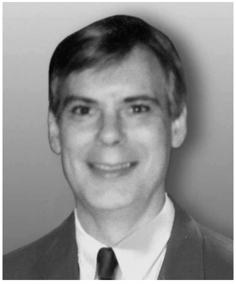

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